IFFERENTIAL GEOMETRY
OF THREE DIMENSIONS

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CHAPTER I

DIFFERENTIAL INVARIANTS FOR A SURFACE

1. Gradient of a scalar function. Derivatives. The differential invariants discussed in Chapter XII of the first volume of this work play an important part in the following pages. We shall therefore remind the reader of their chief properties, and collect for reference the most important formulae, of which constant use will be made.

The *gradient* of a scalar point-function ϕ on a given surface S is denoted by $\nabla\phi$. It is a vector quantity whose direction at any point P is that direction on the surface which gives the maximum arc-rate of increase of ϕ, and whose magnitude is this maximum rate of increase. It is thus a vector point-function on the surface. The *derivative*, or rate of increase, of ϕ in any direction on the surface is the resolved part of $\nabla\phi$ in this direction. Thus if $\mathbf{c}$ is a unit surface vector at P, that is to say a unit vector parallel to the tangent plane at P, the derivative of ϕ in the direction of $\mathbf{c}$ has the value $\mathbf{c}\cdot\nabla\phi$. In terms of any convenient parameters u, v on the surface, and the corresponding first order magnitudes E, F, G, the gradient of ϕ may be expressed*

$$\nabla\phi = H^{-2}[(G\phi_1 - F\phi_2)\,\mathbf{r}_1 + (E\phi_2 - F\phi_1)\,\mathbf{r}_2] \quad \text{......(1)},$$

where $\mathbf{r}$ is the position vector of the current point P on the surface, and suffixes 1, 2 denote partial differentiations with respect to u and v respectively. Thus $\nabla\phi$ may be regarded as the result obtained by operating on the function ϕ with the vectorial differential operator

$$\nabla = \frac{1}{H^2}\left[\mathbf{r}_1\left(G\frac{\partial}{\partial u} - F\frac{\partial}{\partial v}\right) + \mathbf{r}_2\left(E\frac{\partial}{\partial v} - F\frac{\partial}{\partial u}\right)\right] \text{......(2)}$$

beying the distributive law. When the parametric curves are rthogonal, $F=0$ and $H^2 = EG$; so that (2) takes the simpler orm

$$\nabla = \frac{1}{E}\mathbf{r}_1\frac{\partial}{\partial u} + \frac{1}{G}\mathbf{r}_2\frac{\partial}{\partial v} \quad \text{....................(3)}.$$

* *Loc. cit.*, Art. 114.

For any choice of parametric curves the operator ∇ and the function $\nabla\phi$ may be expressed very simply by means of the vectors of the system reciprocal to $\mathbf{r}_1, \mathbf{r}_2, \mathbf{n}$. This system $\mathbf{l}, \mathbf{m}, \mathbf{n}$ is defined by

$$H\mathbf{l} = \mathbf{r}_2 \times \mathbf{n}, \quad H\mathbf{m} = \mathbf{n} \times \mathbf{r}_1, \quad H\mathbf{n} = \mathbf{r}_1 \times \mathbf{r}_2.$$

From these relations it follows that

$$\left.\begin{aligned} H^2\mathbf{l} &= \mathbf{r}_2 \times (\mathbf{r}_1 \times \mathbf{r}_2) = G\mathbf{r}_1 - F\mathbf{r}_2 \\ H^2\mathbf{m} &= (\mathbf{r}_1 \times \mathbf{r}_2) \times \mathbf{r}_1 = E\mathbf{r}_2 - F\mathbf{r}_1 \end{aligned}\right\},$$

and consequently the gradient of ϕ is given by

$$\nabla\phi = \mathbf{l}\frac{\partial\phi}{\partial u} + \mathbf{m}\frac{\partial\phi}{\partial v} \quad \text{......................(4)}$$

and the operator ∇ by

$$\nabla = \mathbf{l}\frac{\partial}{\partial u} + \mathbf{m}\frac{\partial}{\partial v} \quad \text{........................(5).}$$

If θ is a point-function on S, and $f(\theta)$ a function of θ, it follows immediately from the definition by (1) or (4) that

$$\nabla f(\theta) = f'(\theta)\nabla\theta \quad \text{......................(6).}$$

Similarly, if $f(\theta, \phi, \psi, \ldots)$ is a function of several point-functions $\theta, \phi, \psi, \ldots$,

$$\nabla f(\theta, \phi, \ldots) = \frac{\partial f}{\partial\theta}\nabla\theta + \frac{\partial f}{\partial\phi}\nabla\phi + \ldots \quad \text{.........(7).}$$

The operator $\mathbf{c}\cdot\nabla$ may also be applied to a vector point-function $\mathbf{u}$, giving the derivative of $\mathbf{u}$ in the direction of the unit vector $\mathbf{c}$. Thus by (5)

$$\mathbf{c}\cdot\nabla\mathbf{u} = (\mathbf{c}\cdot\mathbf{l})\frac{\partial\mathbf{u}}{\partial u} + (\mathbf{c}\cdot\mathbf{m})\frac{\partial\mathbf{u}}{\partial v}.$$

And, though the interpretation of $\mathbf{c}\cdot\nabla\mathbf{u}$ as the rate of change of $\mathbf{u}$ in the direction of $\mathbf{c}$ is applicable only when $\mathbf{c}$ is tangential to the surface, and a unit vector, we define the function $\mathbf{c}\cdot\nabla\mathbf{u}$ for all values of $\mathbf{c}$ by the above equation. In particular, if $\mathbf{c}$ is normal to the surface, $\mathbf{c}\cdot\nabla\mathbf{u}$ is equal to zero.

2. Divergence and rotation of a vector. The operator ∇ may be applied to a vector function $\mathbf{u}$ in different ways. One of these gives a scalar function called the *divergence* of $\mathbf{u}$, and denoted by div $\mathbf{u}$ or $\nabla\cdot\mathbf{u}$. It is defined by

$$\begin{aligned} \operatorname{div}\mathbf{u} &= \mathbf{l}\cdot\mathbf{u}_1 + \mathbf{m}\cdot\mathbf{u}_2 \\ &= H^{-2}[\mathbf{r}_1\cdot(G\mathbf{u}_1 - F\mathbf{u}_2) + \mathbf{r}_2\cdot(E\mathbf{u}_2 - F\mathbf{u}_1)] \quad \text{......(8),} \end{aligned}$$

and is invariant with respect to the choice of parameters u, v. Similarly ∇ may be applied to $\mathbf{u}$ in such a way as to give a vector differential invariant, called the *rotation* or *curl* of $\mathbf{u}$, and denoted by rot $\mathbf{u}$, curl $\mathbf{u}$ or $\nabla \times \mathbf{u}$. It is defined by

$$\begin{aligned}\operatorname{rot} \mathbf{u} &= \mathbf{l} \times \mathbf{u}_1 + \mathbf{m} \times \mathbf{u}_2 \\ &= H^{-2} [\mathbf{r}_1 \times (G\mathbf{u}_1 - F\mathbf{u}_2) + \mathbf{r}_2 \times (E\mathbf{u}_2 - F\mathbf{u}_1)] \quad \text{......(9).}\end{aligned}$$

We shall use the abbreviation rot $\mathbf{u}$, rather than curl $\mathbf{u}$, throughout this volume.

The unit vector $\mathbf{n}$, normal to the surface, is a point-function which plays an important part. From the above formulae it is easily verified, as we have already shown*, that

$$\operatorname{div} \mathbf{n} = -J, \quad \operatorname{rot} \mathbf{n} = 0 \quad \text{..................(10),}$$

J being the first curvature (or mean curvature) of the surface. Similarly for any vector $\phi\mathbf{n}$, normal to the surface, we find

$$\operatorname{div} \phi\mathbf{n} = -J\phi, \quad \operatorname{rot} \phi\mathbf{n} = \nabla\phi \times \mathbf{n} \quad \text{............(11),}$$

while, for any tangential vector $P\mathbf{r}_1 + Q\mathbf{r}_2$, we have

$$\operatorname{div} (P\mathbf{r}_1 + Q\mathbf{r}_2) = \frac{1}{H} \left[\frac{\partial}{\partial u} (HP) + \frac{\partial}{\partial v} (HQ) \right] \quad \text{......(12).}$$

In particular, if $\mathbf{a}$ and $\mathbf{b}$ are the unit tangents to the orthogonal parametric curves v = const. and u = const. respectively,

$$\operatorname{div} \mathbf{a} = \frac{G_1}{2G\sqrt{E}}, \quad \operatorname{div} \mathbf{b} = \frac{E_2}{2E\sqrt{G}} \quad \text{.........(13).}$$

These two quantities will be met very frequently in the following pages.

3. Formulae of expansion. Let ϕ, ψ be scalar functions and $\mathbf{u}$, $\mathbf{v}$ vector functions on the given surface. We frequently require expansions for the gradients of the products $\phi\psi$ and $\mathbf{u} \cdot \mathbf{v}$, and for the divergence and rotation of the vectors $\phi\mathbf{u}$ and $\mathbf{u} \times \mathbf{v}$, in terms of differential invariants of the separate functions. From (1) or (4), and the rule for differentiating a product, it follows immediately that

$$\nabla (\phi\psi) = \phi\nabla\psi + \psi\nabla\phi \quad \text{..................(14).}$$

* *Loc. cit.*, Arts. 116, 118.

Similarly, as we have already shown*, it follows from (8) and (9) that

$$\operatorname{div}(\phi\mathbf{u}) = \nabla\phi \cdot \mathbf{u} + \phi \operatorname{div}\mathbf{u} \quad \text{.............}(15),$$

$$\operatorname{rot}(\phi\mathbf{u}) = \nabla\phi \times \mathbf{u} + \phi \operatorname{rot}\mathbf{u} \quad \text{.............}(16),$$

$$\operatorname{div}(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot\operatorname{rot}\mathbf{u} - \mathbf{u}\cdot\operatorname{rot}\mathbf{v} \quad \text{...........}(17).$$

There are two other relations, not previously considered, which we shall have frequent occasion to use. These are expressed by the formulae

$$\operatorname{rot}(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot\nabla\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{v} + \mathbf{u}\operatorname{div}\mathbf{v} - \mathbf{v}\operatorname{div}\mathbf{u} \quad \text{......}(18),$$

$$\nabla(\mathbf{u}\cdot\mathbf{v}) = \mathbf{v}\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{v} + \mathbf{v}\times\operatorname{rot}\mathbf{u} + \mathbf{u}\times\operatorname{rot}\mathbf{v} \quad (19).$$

To prove the first of these we observe that, in virtue of (9),

$$\begin{aligned}\operatorname{rot}(\mathbf{u}\times\mathbf{v}) &= \mathbf{l}\times(\mathbf{u}_1\times\mathbf{v}+\mathbf{u}\times\mathbf{v}_1) + \mathbf{m}\times(\mathbf{u}_2\times\mathbf{v}+\mathbf{u}\times\mathbf{v}_2)\\ &= (\mathbf{l}\cdot\mathbf{v})\,\mathbf{u}_1 - (\mathbf{l}\cdot\mathbf{u}_1)\,\mathbf{v} + (\mathbf{l}\cdot\mathbf{v}_1)\,\mathbf{u} - (\mathbf{l}\cdot\mathbf{u})\,\mathbf{v}_1\\ &\quad + (\mathbf{m}\cdot\mathbf{v})\,\mathbf{u}_2 - (\mathbf{m}\cdot\mathbf{u}_2)\,\mathbf{v} + (\mathbf{m}\cdot\mathbf{v}_2)\,\mathbf{u} - (\mathbf{m}\cdot\mathbf{u})\,\mathbf{v}_2\\ &= \mathbf{v}\cdot\nabla\mathbf{u} - \mathbf{v}\operatorname{div}\mathbf{u} + \mathbf{u}\operatorname{div}\mathbf{v} - \mathbf{u}\cdot\nabla\mathbf{v},\end{aligned}$$

which is the required formula (18). To demonstrate (19) we have similarly

$$\nabla(\mathbf{u}\cdot\mathbf{v}) = \mathbf{l}\,(\mathbf{u}_1\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{v}_1) + \mathbf{m}\,(\mathbf{u}_2\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{v}_2) \quad \text{......(i)}.$$

Also, in virtue of (9),

$$\begin{aligned}\mathbf{v}\times\operatorname{rot}\mathbf{u} &= \mathbf{v}\times(\mathbf{l}\times\mathbf{u}_1 + \mathbf{m}\times\mathbf{u}_2)\\ &= (\mathbf{v}\cdot\mathbf{u}_1)\,\mathbf{l} - (\mathbf{v}\cdot\mathbf{l})\,\mathbf{u}_1 + (\mathbf{v}\cdot\mathbf{u}_2)\,\mathbf{m} - (\mathbf{v}\cdot\mathbf{m})\,\mathbf{u}_2\\ &= (\mathbf{v}\cdot\mathbf{u}_1)\,\mathbf{l} + (\mathbf{v}\cdot\mathbf{u}_2)\,\mathbf{m} - \mathbf{v}\cdot\nabla\mathbf{u} \quad \text{..............(ii)},\end{aligned}$$

and similarly

$$\mathbf{u}\times\operatorname{rot}\mathbf{v} = (\mathbf{u}\cdot\mathbf{v}_1)\,\mathbf{l} + (\mathbf{u}\cdot\mathbf{v}_2)\,\mathbf{m} - \mathbf{u}\cdot\nabla\mathbf{v} \quad \text{......(iii)}.$$

From (i), (ii) and (iii) the formula (19) follows at once.

An important particular case of (19) is that in which the two vectors $\mathbf{u}$, $\mathbf{v}$ are equal and of constant length. Then

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{u}^2 = \text{const.},$$

and the first member of (19) vanishes. Hence, *if* $\mathbf{u}$ *is a vector of constant length,*

$$\mathbf{u}\cdot\nabla\mathbf{u} = -\,\mathbf{u}\times\operatorname{rot}\mathbf{u} \quad \text{.................}(20).$$

* *Loc. cit.*, Art. 120.

Example. Deduce the Circulation Theorem from the Divergence Theorem (Vol. I, Arts. 122, 124).

With the notation of those theorems let C be a closed curve on the given surface, ds the length of an element of arc, $\mathbf{t}$ the unit tangent and $\mathbf{m}$ the unit surface vector perpendicular to $\mathbf{t}$ and drawn outward, so that $\mathbf{m}, \mathbf{t}, \mathbf{n}$ form a right-handed system. Apply the Divergence Theorem to the vector $\mathbf{u}\times\mathbf{n}$, where $\mathbf{u}$ is a vector point-function for the surface. Then

$$\iint \operatorname{div}(\mathbf{u}\times\mathbf{n})\,dS = \int_{\circ} (\mathbf{u}\times\mathbf{n})\cdot\mathbf{m}\,ds \quad \text{.................(i)},$$

the line integral being taken round the curve C, and the surface integral over the region enclosed. Now

$$(\mathbf{u}\times\mathbf{n})\cdot\mathbf{m} = \mathbf{u}\cdot(\mathbf{n}\times\mathbf{m}) = \mathbf{u}\cdot\mathbf{t}.$$

Also, in virtue of (17), $\qquad \operatorname{div}(\mathbf{u}\times\mathbf{n}) = \mathbf{n}\cdot\operatorname{rot}\mathbf{u},$

since $\operatorname{rot}\mathbf{n}$ vanishes identically. Hence we may write (i)

$$\iint \mathbf{n}\cdot\operatorname{rot}\mathbf{u}\,dS = \int_{\circ} \mathbf{u}\cdot\mathbf{t}\,ds = \int_{\circ} \mathbf{u}\cdot d\mathbf{r} \quad \text{.................(ii)},$$

which expresses the Circulation Theorem for the function $\mathbf{u}$ and the curve C, the line integral in (ii) being the circulation of the vector $\mathbf{u}$ round the curve C.

4. Differential invariants of the second order. Certain differential invariants of the second order are of frequent occurrence. The most important of these is the divergence of the gradient of a scalar function. This function $\operatorname{div}\operatorname{grad}\phi$ or $\nabla\cdot\nabla\phi$ will be denoted by $\nabla^2\phi$. It is identical with Beltrami's "differential parameter of the second order." From (1) and (12) it follows that

$$\nabla^2\phi = \frac{1}{H}\left[\frac{\partial}{\partial u}\left(\frac{G\phi_1 - F\phi_2}{H}\right) + \frac{\partial}{\partial v}\left(\frac{E\phi_2 - F\phi_1}{H}\right)\right] \quad \text{...(21)},$$

or, when the parametric curves are orthogonal,

$$\nabla^2\phi = \frac{1}{\sqrt{(EG)}}\left[\frac{\partial}{\partial u}\left(\phi_1\sqrt{\frac{G}{E}}\right) + \frac{\partial}{\partial v}\left(\phi_2\sqrt{\frac{E}{G}}\right)\right] \quad \text{...(22)}.$$

The operator ∇^2, defined by these equations, may also be applied › a vector point-function, giving a vector differential invariant of ıe second order. We have already shown that*, if $\mathbf{r}$ is the posi- ɔn vector of the current point on the surface, and $\mathbf{n}$ the unit ɔrmal,

$$\nabla^2\mathbf{r} = J\mathbf{n},$$

nd

$$\nabla^2\mathbf{n} = -(J^2 - 2K)\mathbf{n} - \nabla J \quad \text{..............(23)},$$

* *Loc. cit.*, Art. 119.

K being the second curvature (or Gaussian curvature) of the surface. From the last equation we deduce the important formula

$$2K = \mathbf{n} \cdot \nabla^2 \mathbf{n} + (\operatorname{div} \mathbf{n})^2 \quad \text{.................(24)},$$

which expresses K as a differential invariant of **n** *of the second order.*

The rotation of the gradient of ϕ, or rot $\nabla\phi$, is frequently met with; and we have shown that this vector is tangential to the surface*. Similarly the divergence of the rotation of **u**, or div rot **u**, also presents itself. If **u** is normal to the surface this function vanishes everywhere. For any normal vector is of the form $\phi\mathbf{n}$, and by (16) and (17), since rot **n** is zero, we have

$$\operatorname{div} \operatorname{rot} \phi\mathbf{n} = \operatorname{div} (\nabla\phi \times \mathbf{n}) = \mathbf{n} \cdot (\operatorname{rot} \nabla\phi),$$

which vanishes, since rot $\nabla\phi$ is a surface vector.

Ex. 1. Prove the formulae

$$\nabla^2 (\phi\psi) = \phi\nabla^2\psi + 2\nabla\phi \cdot \nabla\psi + \psi\nabla^2\phi,$$
$$\operatorname{rot} \nabla (\phi\psi) = \phi \operatorname{rot} \nabla\psi + \psi \operatorname{rot} \nabla\phi.$$

Ex. 2. If θ is a point-function, and $f(\theta)$ a function of θ,

$$\nabla^2 f(\theta) = f''(\theta)(\nabla\theta)^2 + f'(\theta)\nabla^2\theta.$$

5. Order of directional differentiations. If a function is differentiated successively in two different directions, the order of the differentiations is not in general commutative. Let ϕ be the function considered, and let the directions of differentiation be those of the parametric curves. If $\frac{d}{ds}$ and $\frac{d}{ds'}$ be used for the moment as symbols of differentiation in the directions of $v = \text{const.}$ and $u = \text{const.}$ respectively, we have

$$\frac{d\phi}{ds} = \frac{1}{\sqrt{E}} \frac{\partial\phi}{\partial u},$$

and therefore

$$\frac{d^2\phi}{ds'ds} = \frac{1}{\sqrt{G}} \left(\frac{1}{\sqrt{E}} \frac{\partial^2\phi}{\partial v \partial u} - \frac{E_2}{2E\sqrt{E}} \frac{\partial\phi}{\partial u} \right).$$

Similarly $$\frac{d^2\phi}{ds ds'} = \frac{1}{\sqrt{E}} \left(\frac{1}{\sqrt{G}} \frac{\partial^2\phi}{\partial u \partial v} - \frac{G_1}{2G\sqrt{G}} \frac{\partial\phi}{\partial v} \right).$$

* *Loc. cit.*, Art. 118.

Consequently the difference of these two derivatives has a value given by

$$\frac{d^2\phi}{ds\,ds'} - \frac{d^2\phi}{ds'ds} = \frac{E_2}{2E\sqrt{G}}\frac{d\phi}{ds} - \frac{G_1}{2G\sqrt{E}}\frac{d\phi}{ds'} \quad \text{......(25).}$$

A case of particular importance is that in which the two directions are at right angles. Suppose then that the parametric curves are orthogonal, with **a** and **b** as their unit tangents at any point. Then, in virtue of (13), we may write the above equation

$$\mathbf{a}\cdot\nabla(\mathbf{b}\cdot\nabla\phi) - \mathbf{b}\cdot\nabla(\mathbf{a}\cdot\nabla\phi) = (\mathbf{a}\cdot\nabla\phi)\operatorname{div}\mathbf{b} - (\mathbf{b}\cdot\nabla\phi)\operatorname{div}\mathbf{a}.$$

Now the geodesic curvatures γ, γ' of the orthogonal curves $v = \text{const.}$, $u = \text{const.}$ are given by*

$$\gamma = -\operatorname{div}\mathbf{b}, \quad \gamma' = \operatorname{div}\mathbf{a},$$

in virtue of (13). Consequently the difference of the two derivatives given above may be expressed

$$\begin{aligned}\frac{d^2\phi}{ds'ds} - \frac{d^2\phi}{ds\,ds'} &= \gamma\mathbf{a}\cdot\nabla\phi + \gamma'\mathbf{b}\cdot\nabla\phi \\ &= (\gamma\mathbf{a} + \gamma'\mathbf{b})\cdot\nabla\phi \quad \text{..............(26).}\end{aligned}$$

6. Derivatives of the unit vectors a, b, n. As above let **a**, **b** be the unit tangents to the orthogonal parametric curves. It will be found convenient to record for reference the derivatives of **a**, **b**, **n** in the directions of **a** and **b**. Using the values of the partial derivatives of **a**, **b** with respect to u and v, as found in Vol. I, Art. 41, we have

$$\frac{1}{\sqrt{E}}\frac{\partial\mathbf{a}}{\partial u} = \frac{L}{E}\mathbf{n} - \frac{E_2}{2E\sqrt{G}}\mathbf{b},$$

$$\frac{1}{\sqrt{G}}\frac{\partial\mathbf{a}}{\partial v} = \frac{M}{\sqrt{(EG)}}\mathbf{n} + \frac{G_1}{2G\sqrt{E}}\mathbf{b},$$

which may be expressed

$$\left.\begin{aligned}\mathbf{a}\cdot\nabla\mathbf{a} &= \kappa_n\mathbf{n} + \gamma\mathbf{b} \\ \mathbf{b}\cdot\nabla\mathbf{a} &= \tau\mathbf{n} + \gamma'\mathbf{b}\end{aligned}\right\} \quad \text{.....................(27),}$$

where γ, γ' are the geodesic curvatures of the curves $v = \text{const.}$ and $u = \text{const.}$ respectively, κ_n the normal curvature in the direction

* See also Vol. I, Art. 121.

of the former, and τ the torsion of the geodesic tangent in the same direction. Similarly we find for the derivatives of $\mathbf{b}$

$$\left.\begin{aligned}\mathbf{a}\cdot\nabla\mathbf{b} &= \tau\mathbf{n} \quad -\gamma\mathbf{a}\\ \mathbf{b}\cdot\nabla\mathbf{b} &= \kappa_n'\mathbf{n}-\gamma'\mathbf{a}\end{aligned}\right\} \qquad \text{..................(28)},$$

κ_n' being the normal curvature in the direction of $\mathbf{b}$.

Again, when the parametric curves are orthogonal, the derivatives of $\mathbf{n}$ found in Vol. I, Art. 27, become

$$\frac{\partial\mathbf{n}}{\partial u}=-\frac{L}{E}\frac{\partial\mathbf{r}}{\partial u}-\frac{M}{G}\frac{\partial\mathbf{r}}{\partial v},$$

$$\frac{\partial\mathbf{n}}{\partial v}=-\frac{M}{E}\frac{\partial\mathbf{r}}{\partial u}-\frac{N}{G}\frac{\partial\mathbf{r}}{\partial v}.$$

And from these it follows immediately that

$$\left.\begin{aligned}\mathbf{a}\cdot\nabla\mathbf{n} &= -\kappa_n\mathbf{a}-\tau\mathbf{b}\\ \mathbf{b}\cdot\nabla\mathbf{n} &= -\tau\mathbf{a} \quad -\kappa_n'\mathbf{b}\end{aligned}\right\} \qquad \text{..................(29)}.$$

The above formulae will be used very frequently in the following chapters.

***7. Other differential invariants.** In closing this chapter we may draw attention to certain other differential invariants. We shall first prove the theorem:

For any vector point-function on a given surface, the vector product of its derivatives in any two surface directions, divided by the scalar triple product of the unit normal and the unit vectors in those two directions, is an invariant†.

Let $\mathbf{s}$ be a vector function, and $\mathbf{c}$ and $\mathbf{d}$ unit vectors tangential to the surface. Then if $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ are the reciprocal system of vectors to $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{n}$ as defined in Art. 1, the derivatives of $\mathbf{s}$ in the directions of $\mathbf{c}$ and $\mathbf{d}$ are

$$\mathbf{c}\cdot\nabla\mathbf{s}=(\mathbf{c}\cdot\mathbf{l})\,\mathbf{s}_1+(\mathbf{c}\cdot\mathbf{m})\,\mathbf{s}_2,$$

$$\mathbf{d}\cdot\nabla\mathbf{s}=(\mathbf{d}\cdot\mathbf{l})\,\mathbf{s}_1+(\mathbf{d}\cdot\mathbf{m})\,\mathbf{s}_2,$$

and their vector product has the value

$$\begin{aligned}&\{(\mathbf{c}\cdot\mathbf{l})(\mathbf{d}\cdot\mathbf{m})-(\mathbf{c}\cdot\mathbf{m})(\mathbf{d}\cdot\mathbf{l})\}\,\mathbf{s}_1\times\mathbf{s}_2\\ &\qquad=(\mathbf{c}\times\mathbf{d})\cdot(\mathbf{l}\times\mathbf{m})\,\mathbf{s}_1\times\mathbf{s}_2=\frac{1}{H}(\mathbf{c}\times\mathbf{d}\cdot\mathbf{n})\,\mathbf{s}_1\times\mathbf{s}_2.\end{aligned}$$

† See Art. 4 of a paper by the author "On Families of Curves and Surfaces," *Quarterly Journal*, Vol. 50 (1927), pp. 350–361.

Consequently the quotient of this vector product by $(\mathbf{c} \times \mathbf{d}) \cdot \mathbf{n}$ has the value $\mathbf{s}_1 \times \mathbf{s}_2/H$. It is therefore independent of the directions of $\mathbf{c}$ and $\mathbf{d}$, and is thus an invariant. The result may be expressed in the form:

The function $\mathbf{s}_1 \times \mathbf{s}_2/H$ is independent of the choice of parametric curves.

Let us denote this differential invariant of $\mathbf{s}$ by $\Lambda\mathbf{s}$. A simple case already met with is that in which $\mathbf{s} = \mathbf{r}$. For

$$\Lambda\mathbf{r} = \mathbf{r}_1 \times \mathbf{r}_2/H = \mathbf{n}.$$

Also, the same invariant of the unit normal $\mathbf{n}$ is equal to $K\mathbf{n}$, where K is the second curvature of the surface. For, taking the lines of curvature as parametric curves, we have

$$\begin{aligned}\Lambda\mathbf{n} &= \frac{\mathbf{n}_1 \times \mathbf{n}_2}{H} = \frac{1}{H}\left(-\frac{L}{E}\mathbf{r}_1\right) \times \left(-\frac{N}{G}\mathbf{r}_2\right) \\ &= K\mathbf{r}_1 \times \mathbf{r}_2/H = K\mathbf{n} \qquad \text{.......................(30).}\end{aligned}$$

Consequently the second curvature of the surface may be expressed as a differential invariant of the unit normal in the form

$$K = \mathbf{n} \cdot (\Lambda\mathbf{n}) \qquad \text{.......................(31).}$$

And this may be written in terms of $\mathbf{r}$,

$$K = (\Lambda\mathbf{r}) \cdot (\Lambda\Lambda\mathbf{r}) \qquad \text{....................(32).}$$

Or again, if ϕ is a point-function for the surface, so also is $\phi\mathbf{n}$. Then

$$\begin{aligned}\Lambda(\phi\mathbf{n}) &= \frac{1}{H}(\phi_1\mathbf{n} + \phi\mathbf{n}_1) \times (\phi_2\mathbf{n} + \phi\mathbf{n}_2) \\ &= \phi^2 K\mathbf{n} + \frac{1}{H}\phi\mathbf{n} \times (\phi_1\mathbf{n}_2 - \phi_2\mathbf{n}_1).\end{aligned}$$

The first term is itself an invariant; and, since $\mathbf{n}$ is perpendicular to $\mathbf{n}_1$ and $\mathbf{n}_2$, we have the result:

The function $(\phi_1\mathbf{n}_2 - \phi_2\mathbf{n}_1)/H$ is independent of the choice of parametric curves.

Similarly by considering $\Lambda(\phi\mathbf{s})$ we may show that

$$\mathbf{s} \times (\phi_1\mathbf{s}_2 - \phi_2\mathbf{s}_1)/H$$

is a differential invariant of ϕ and $\mathbf{s}$.

EXAMPLES I

1. Show that, for orthogonal parametric curves,

$$\begin{aligned}\operatorname{rot}\mathbf{a}&=\frac{M}{\sqrt{(EG)}}\mathbf{a}-\frac{L}{E}\mathbf{b}-\frac{E_2}{2E\sqrt{G}}\mathbf{n}\\&=\tau\mathbf{a}-\kappa_n\mathbf{b}+\gamma\mathbf{n},\\ \operatorname{rot}\mathbf{b}&=\frac{N}{G}\mathbf{a}-\frac{M}{\sqrt{(EG)}}\mathbf{b}+\frac{G_1}{2G\sqrt{E}}\mathbf{n}\\&=\kappa_n'\mathbf{a}-\tau\mathbf{b}+\gamma'\mathbf{n},\end{aligned}$$

and deduce the formula

$$K=\mathbf{n}\cdot(\operatorname{rot}\mathbf{a}\times\operatorname{rot}\mathbf{b})=[\mathbf{n},\operatorname{rot}\mathbf{a},\operatorname{rot}\mathbf{b}].$$

2. Show that, for any choice of parametric curves,

$$\operatorname{rot}(P\mathbf{r}_1+Q\mathbf{r}_2)=\frac{1}{H}\left[\frac{\partial}{\partial u}(FP+GQ)-\frac{\partial}{\partial v}(EP+FQ)\right]\mathbf{n}$$
$$+\frac{1}{H}[(PM+QN)\,\mathbf{r}_1-(PL+QM)\,\mathbf{r}_2],$$

and deduce that rot $\nabla\phi$ is tangential to the surface.

3. If $f(\theta,\phi)$ is a function of the two point-functions θ, ϕ, show that

$$\nabla^2 f=\frac{\partial f}{\partial\theta}\nabla^2\theta+\frac{\partial f}{\partial\phi}\nabla^2\phi+\frac{\partial^2 f}{\partial\theta^2}(\nabla\theta)^2+2\frac{\partial^2 f}{\partial\theta\partial\phi}\nabla\theta\cdot\nabla\phi+\frac{\partial^2 f}{\partial\phi^2}(\nabla\phi)^2,$$

and that

$$(\nabla f)^2=\left(\frac{\partial f}{\partial\theta}\right)^2(\nabla\theta)^2+2\frac{\partial f}{\partial\theta}\frac{\partial f}{\partial\phi}\nabla\theta\cdot\nabla\phi+\left(\frac{\partial f}{\partial\phi}\right)^2(\nabla\phi)^2.$$

4. Prove the relations

$$\operatorname{div}\operatorname{rot}\phi\mathbf{s}=\mathbf{s}\cdot(\operatorname{rot}\nabla\phi)+\phi\operatorname{div}\operatorname{rot}\mathbf{s},$$
$$\nabla^2(\phi\mathbf{s})=\phi\nabla^2\mathbf{s}+2\nabla\phi\cdot\nabla\mathbf{s}+\mathbf{s}\nabla^2\phi.$$

5. If $\mathbf{a}$ and $\mathbf{b}$ have the usual significance, show that

$$K=-\operatorname{div}(\mathbf{a}\operatorname{div}\mathbf{a}+\mathbf{b}\operatorname{div}\mathbf{b}),$$

and that $$K=(\mathbf{a}\cdot\nabla\mathbf{a})\cdot(\mathbf{b}\cdot\nabla\mathbf{b})-(\mathbf{b}\cdot\nabla\mathbf{a})\cdot(\mathbf{a}\cdot\nabla\mathbf{b}).$$

6. With the same notation, show that

$$\begin{aligned}\mathbf{n}\cdot\operatorname{rot}(\mathbf{a}\operatorname{div}\mathbf{a}+\mathbf{b}\operatorname{div}\mathbf{b})&=\operatorname{div}\{(\mathbf{a}\operatorname{div}\mathbf{a}+\mathbf{b}\operatorname{div}\mathbf{b})\times\mathbf{n}\}\\&=\operatorname{div}(\mathbf{a}\operatorname{div}\mathbf{b}-\mathbf{b}\operatorname{div}\mathbf{a})\\&=\frac{1}{2\sqrt{(EG)}}\frac{\partial^2}{\partial u\partial v}\log\frac{E}{G}.\end{aligned}$$

Hence deduce that, if the orthogonal parametric curves constitute an isometric system, the vector $\mathbf{a}\operatorname{div}\mathbf{a}+\mathbf{b}\operatorname{div}\mathbf{b}$ is the gradient of some scalar function.

Also show that $$\mathbf{a}\cdot\nabla\operatorname{div}\mathbf{b}-\mathbf{b}\cdot\nabla\operatorname{div}\mathbf{a}=0$$
is a necessary and sufficient condition that the parametric curves be isometric.

7. Prove the relation

$$\mathbf{n} \cdot \operatorname{rot}(\mathbf{n} \times \nabla\theta) = \nabla^2\theta,$$

and show that $$\operatorname{div}(\mathbf{n} \times \nabla\theta) = 0.$$

8. If $\mathbf{t}$ and $\mathbf{t}'$ are defined by

$$\mathbf{t} = \mathbf{a}\cos\theta + \mathbf{b}\sin\theta, \quad \mathbf{t}' = \mathbf{b}\cos\theta - \mathbf{a}\sin\theta,$$

show that
$$\operatorname{div}\mathbf{t} = \cos\theta \operatorname{div}\mathbf{a} + \sin\theta \operatorname{div}\mathbf{b} + \mathbf{t}' \cdot \nabla\theta,$$
$$\operatorname{div}\mathbf{t}' = \cos\theta \operatorname{div}\mathbf{b} - \sin\theta \operatorname{div}\mathbf{a} - \mathbf{t} \cdot \nabla\theta,$$

and deduce that, if $\mathbf{a}$, $\mathbf{b}$ have their usual significance,

$$\mathbf{t} \operatorname{div}\mathbf{t} + \mathbf{t}' \operatorname{div}\mathbf{t}' = \mathbf{a} \operatorname{div}\mathbf{a} + \mathbf{b} \operatorname{div}\mathbf{b} - \mathbf{n} \times \nabla\theta.$$

9. Considering the gradient of both members of the identity $\mathbf{n} \cdot \nabla\phi = 0$, prove that

$$\nabla\phi \cdot \nabla\mathbf{n} = -\mathbf{n} \times \operatorname{rot}\nabla\phi.$$

Similarly, for any surface vector $\mathbf{t}$,

$$\mathbf{t} \cdot \nabla\mathbf{n} = -\mathbf{n} \times \operatorname{rot}\mathbf{t}.$$

10. From the identity

$$\operatorname{rot}(\mathbf{a} \times \mathbf{b}) = \operatorname{rot}\mathbf{n} = 0,$$

by expanding the first expression, prove that

$$(\mathbf{b} \cdot \nabla\mathbf{a}) \cdot \mathbf{b} = \operatorname{div}\mathbf{a},$$
$$(\mathbf{a} \cdot \nabla\mathbf{b}) \cdot \mathbf{a} = \operatorname{div}\mathbf{b}.$$

Deduce the same relations by expanding the first member of the identity $\nabla(\mathbf{a} \cdot \mathbf{b}) = 0$.

11. Prove the relations

$$\mathbf{b} \cdot \nabla^2\mathbf{a} = \frac{1}{\sqrt{(EG)}}\left(-JM - \frac{1}{2}\frac{\partial^2}{\partial u\,\partial v}\log\frac{E}{G}\right),$$

and $$\mathbf{a} \cdot \nabla^2\mathbf{a} = -(\operatorname{div}\mathbf{a})^2 - (\operatorname{rot}\mathbf{a})^2.$$

12. From the formula $J = -\operatorname{div}\mathbf{n}$, show that

$$J = \mathbf{a} \cdot \operatorname{rot}\mathbf{b} - \mathbf{b} \cdot \operatorname{rot}\mathbf{a}.$$

13. Show that the resolved part of $\operatorname{rot}\operatorname{rot}(\theta\mathbf{n})$ normal to the surface is $-\nabla^2\theta$.

14. If u, v are geodesic polar coordinates, so that $E=1$, $F=0$ and $H^2=G$, show that

$$\nabla\phi = \phi_1\mathbf{r}_1 + \frac{1}{G}\phi_2\mathbf{r}_2,$$

and $$\nabla^2\phi = \frac{1}{H}\left[\frac{\partial}{\partial u}(H\phi_1) + \frac{\partial}{\partial v}\left(\frac{\phi_2}{H}\right)\right].$$

Hence, if H is a function of u only, show that $\int \frac{du}{H}$ satisfies the equation $\nabla^2\phi = 0$, and that

$$K = -\nabla^2 \log H.$$

15. Prove that, with the notation of Art. 1,

$$\mathbf{l}\cdot\mathbf{r}_1=\mathbf{m}\cdot\mathbf{r}_2=1, \quad \mathbf{l}\cdot\mathbf{r}_2=\mathbf{m}\cdot\mathbf{r}_1=0,$$
$$\mathbf{r}_1=E\mathbf{l}+F\mathbf{m}, \quad \mathbf{r}_2=F\mathbf{l}+G\mathbf{m},$$
$$\mathbf{l}^2=G/H^2, \quad \mathbf{m}^2=E/H^2, \quad \mathbf{l}\cdot\mathbf{m}=-F/H^2,$$
$$(\mathbf{l}\times\mathbf{m})\cdot\mathbf{n}=[\mathbf{l}, \mathbf{m}, \mathbf{n}]=1/H.$$

16. Show that, for any choice of parameters,

$$\phi_1\mathbf{r}_2-\phi_2\mathbf{r}_1=H\mathbf{n}\times\nabla\phi.$$

The second member has the value [Art. 1 (4)]

$$\mathbf{n}\times(\mathbf{r}_2\times\mathbf{n}\phi_1+\mathbf{n}\times\mathbf{r}_1\phi_2)=\phi_1\mathbf{r}_2-\phi_2\mathbf{r}_1.$$

NOTE. The gradient of a scalar function for a given surface was introduced by Burali-Forti in a memoir "Fondamenti per la geometria differenziale, etc.," *Rend. Cir. Mat. Palermo*, t. 33 (1912). Several of the formulae found above in Art. 3 and in Vol. I, Arts. 122, 123, were given by P. Burgatti in a paper entitled "I teoremi del gradiente, della divergenza, della rotazione sopra una superficie, e loro applicazione ai potenziale," *Mem. R. Acc. delle Sc. dell' Inst. di Bologna*, t. 4 (1917), pp. 103–112. Burgatti defined the invariants for a given surface in terms of those for space; and, from the known formulae connecting the latter, deduced the corresponding formulae for surface invariants. His "theorem of divergence" is identical with that of Vol. I, Art. 122, but his "theorem of rotation" is that expressed in Art. 123 (29).

Other interesting results are given by P. C. Delens in his *Méthodes et Problèmes des Géométries Différentielles Euclidienne et Conforme* (Paris, 1927).

CHAPTER II

FAMILIES OF CURVES ON A SURFACE

8. Curvature properties. Consider a singly infinite family of curves on a given surface. Let $\mathbf{t}$ be the unit tangent to the curve at any point, $\mathbf{n}$ the unit normal to the surface and $\mathbf{b}$ the unit surface vector $\mathbf{n}\times\mathbf{t}$, so that $\mathbf{t}$, $\mathbf{b}$, $\mathbf{n}$ form a right-handed system. These vectors are all point-functions for the surface.

The vector curvature of the curve at any point P is the derivative of $\mathbf{t}$ in the direction of the curve. This quantity $\mathbf{t}\cdot\nabla\mathbf{t}$ is also expressible as $-\mathbf{t}\times\operatorname{rot}\mathbf{t}$, by (20) of Art. 3. The *geodesic curvature* of the curve at the point considered is the tangential resolute of the vector curvature, that is to say, the resolved part of the vector curvature in the direction of $\mathbf{b}$. Hence, denoting the geodesic curvature by γ, we have

$$\gamma = -\mathbf{b}\cdot(\mathbf{t}\times\operatorname{rot}\mathbf{t}) = (\mathbf{t}\times\mathbf{b})\cdot\operatorname{rot}\mathbf{t} = \mathbf{n}\cdot\operatorname{rot}\mathbf{t} \quad \text{......(1).}$$

But
$$\operatorname{div}\mathbf{b} = \operatorname{div}(\mathbf{n}\times\mathbf{t}) = -\mathbf{n}\cdot\operatorname{rot}\mathbf{t},$$

so that the geodesic curvature is also given by

$$\gamma = -\operatorname{div}\mathbf{b} \quad \text{..........................(2).}$$

Hence the theorem*:

Given a family of curves on a surface, with $\mathbf{t}$ *as the unit tangent in the assigned positive direction along the curve, the geodesic curvature of a member of the family has the value* $\mathbf{n}\cdot\operatorname{rot}\mathbf{t}$ *or* $-\operatorname{div}(\mathbf{n}\times\mathbf{t})$.

The *normal curvature* κ_n of the surface in the direction of $\mathbf{t}$ is the resolved part of the vector curvature of the curve in the direction of $\mathbf{n}$. Thus

$$\kappa_n = -\mathbf{n}\cdot(\mathbf{t}\times\operatorname{rot}\mathbf{t}) = -\mathbf{b}\cdot\operatorname{rot}\mathbf{t} \quad \text{..............(3).}$$

Consider now the geodesic which touches the curve at the point P. The unit tangent at P to this geodesic is $\mathbf{t}$, its unit principal normal is $\mathbf{n}$, and its unit binormal is therefore $-\mathbf{b}$. In virtue of the Serret-Frenet formulae for a twisted curve, the *torsion of this*

* See also Vol. I, Arts. 121 and 126.

geodesic tangent is the resolved part in the direction of the binormal of the derivative of **n** along the curve. Denoting it by τ we have

$$\tau = -\mathbf{b}\cdot(\mathbf{t}\cdot\nabla\mathbf{n}).$$

But, on taking the gradient of both members of the identity $\mathbf{t}\cdot\mathbf{n}=0$, we find by (19) of Art. 3 that

$$\mathbf{t}\cdot\nabla\mathbf{n} = -\mathbf{n}\times\operatorname{rot}\mathbf{t}.$$

Consequently the above torsion has the value*

$$\tau = \mathbf{b}\cdot(\mathbf{n}\times\operatorname{rot}\mathbf{t}) = \mathbf{t}\cdot\operatorname{rot}\mathbf{t}\quad\text{.................(4)}.$$

Since the geodesic curvature vanishes when the curves are geodesics, the normal curvature when they are asymptotic lines, and the torsion of the geodesic tangent when they are lines of curvature, it follows from (1), (3) and (4) that:

A family of curves with unit tangent **t** *will be geodesics if* $\mathbf{n}\cdot\operatorname{rot}\mathbf{t}$ *vanishes identically; they will be lines of curvature if* $\mathbf{t}\cdot\operatorname{rot}\mathbf{t}$ *is everywhere zero; they will be asymptotic lines if* $\mathbf{b}\cdot\operatorname{rot}\mathbf{t}$ *vanishes identically.*

9. Rate of rotation of the trihedral. The above formulae show that the resolved parts of the vector rot **t** in the directions of **t**, **b** and **n** are τ, $-\kappa_n$ and γ respectively. But the arc-rate of rotation of the trihedral **t**, **b**, **n** as the point P moves along the curve is a vector with the same resolved parts. For the arc-rate of turning of the trihedral about **t** is the torsion of the geodesic tangent; the arc-rate of turning about **b** is the negative of the normal curvature, and the arc-rate of turning about **n** is the tangential curvature of the curve. Consequently:

The vector rot **t** *gives the arc-rate of rotation of the trihedral* **t**, **b**, **n** *as the point of contact moves along a curve of the family.*

From this result it follows directly that the arc-rate of turning of the tangent plane as the point of contact moves along the curve is the component of rot **t** tangential to the surface; for rotation about the normal does not affect the tangent plane. But the direction of this vector is the direction conjugate to that of **t**; for this conjugate direction is the limiting direction of the line of

* See also Vol. I, p. 250.

intersection of the tangent planes at P and a neighbouring point Q on the curve, as Q tends to coincidence with P. Hence:

If $\mathbf{t}$ *is the unit tangent for a family of curves on a surface, the tangential component of* rot $\mathbf{t}$ *gives the direction conjugate to that of* $\mathbf{t}$, *and also the arc-rate of rotation of the tangent plane as the point of contact moves along a curve of the family**.

Consequently, if θ is the angle of rotation in the positive sense from the direction of $\mathbf{t}$ to its conjugate direction,

$$\tan\theta = \frac{\mathbf{b}\cdot\operatorname{rot}\mathbf{t}}{\mathbf{t}\cdot\operatorname{rot}\mathbf{t}} = -\frac{\kappa_n}{\tau} \quad\quad (5).$$

It follows that, for any two conjugate directions, the quotient of the normal curvature by the torsion of the geodesic tangent has opposite values. For a principal direction τ is zero, and the direction is perpendicular to its conjugate. For an asymptotic direction the normal curvature is zero, and the direction is self-conjugate.

In the case of a *family of geodesics* $\mathbf{n}\cdot\operatorname{rot}\mathbf{t}$ vanishes identically, so that rot $\mathbf{t}$ is tangential to the surface. Consequently rot $\mathbf{t}$ gives both the direction conjugate to that of $\mathbf{t}$, and also the arc-rate of rotation of the tangent plane as the point of contact moves along one of the geodesics.

10. Moment of the family. Line of zero moment. The quantity τ has hitherto been regarded as something associated with a certain geodesic, and not as a property of the given family of curves. We shall now show that there is a property of the family

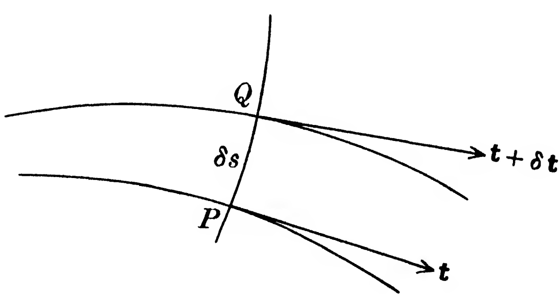

Fig. 1.

* See also Art. 5 of the author's paper "On Families of Curves and Surfaces," *Quarterly Journal*, Vol. 50 (1927), p. 356.

which is represented by τ. Let $\mathbf{t}$ be the unit tangent to the curve at P, and $\mathbf{t}+\delta\mathbf{t}$ that to the curve through an adjacent point Q, at a distance δs from P in the sense of $\mathbf{b}$ along the orthogonal trajectory through P. The mutual moment of these two tangents is the resolved part along $\mathbf{t}$ of the moment of $\mathbf{t}+\delta\mathbf{t}$ about P. This is given, as far as terms of the second order, by

$$\mathbf{t}\cdot\{\mathbf{b}\,\delta s\times(\mathbf{t}+\delta\mathbf{t})\}=\delta s\,(\mathbf{t}\times\mathbf{b})\cdot\delta\mathbf{t}=\delta s\,\mathbf{n}\cdot\delta\mathbf{t}.$$

The limiting value of the quotient of this mutual moment by $(\delta s)^2$, as δs tends to zero, will be called the *moment** of the family of curves for the point P. Denoting it by m we have

$$m=\mathbf{n}\cdot\frac{d\mathbf{t}}{ds}=\mathbf{n}\cdot(\mathbf{b}\cdot\nabla\mathbf{t})=\tau\,\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

by (27) of Art. 6. Thus *the moment of the family of curves at any point is equal to the torsion of the geodesic tangent*; and its value at any point depends only on the direction of $\mathbf{t}$. It vanishes wherever the curve of the family is tangent to a line of curvature. The locus of such points may be called the *line of zero moment* of the family. Its equation is $\mathbf{t}\cdot\operatorname{rot}\mathbf{t}=0$. The moment of a family of lines of curvature vanishes identically. We may also remark in passing that, for a family of generators of a skew surface, the parameter of distribution† of a generator is the reciprocal of the value of the moment of the family at the central point of the generator.

The *line of normal curvature* of the family of curves may be defined as the locus of points at which the tangential curvature is zero. Its equation is $\mathbf{n}\cdot\operatorname{rot}\mathbf{t}=0$. Similarly the *line of tangential curvature* is the locus of points at which the normal curvature vanishes. It is given by $\mathbf{b}\cdot\operatorname{rot}\mathbf{t}=0$.

11. Line of striction. Divergence of the family. Let us next consider the line which bears to a family of curves on any surface the same relation that the line of striction bears to a family of generators on a skew surface. The line of striction was defined‡ as the locus of the limiting positions of the feet of the common perpendiculars to adjacent generators, as these generators tend to coincidence. But this is clearly the locus of the points at which

* See also *Quarterly Journal*, loc. cit., p. 357.
† Vol. I, Art. 68.
‡ Vol. I, Art. 69.

the derivative of $\mathbf{t}$ in the direction of $\mathbf{b}$ is perpendicular to $\mathbf{b}$, the vectors $\mathbf{t}$, $\mathbf{b}$ having the same significance for the family of generators as for the family of curves already considered. Hence the equation defining the line of striction is

$$\mathbf{b}\cdot(\mathbf{b}\cdot\nabla\mathbf{t})=0 \quad\text{.......................(7)}.$$

We take the same property as defining the *line of striction** of a family of curves on any surface. It is thus the locus of points at which $\mathbf{b}\cdot\nabla\mathbf{t}$ is perpendicular to $\mathbf{b}$. But, in virtue of (27) of Art. 6, the equation (7) is equivalent to

$$\operatorname{div}\mathbf{t}=0 \quad\text{..........................(8)}.$$

Hence the result:

For a singly infinite family of curves on a surface, having $\mathbf{t}$ *as unit tangent, the equation of the line of striction may be expressed* $\operatorname{div}\mathbf{t}=0$.

The points in which any one curve cuts the line of striction may be called the *points of striction* of the curve. These correspond to the central point of a generator of a skew surface. The function $\operatorname{div}\mathbf{t}$ may be called the *divergence* of the family of curves. Hence the line of striction is the locus of points at which the divergence of the family is zero. Also, since $\operatorname{div}\mathbf{t}$ is numerically equal to the geodesic curvature of the orthogonal trajectory of the family of curves, the above theorem may be expressed in the form:

The line of striction of a family of curves is the locus of points at which the geodesic curvature of their orthogonal trajectories is zero.

12. Parallels and geodesics. We have seen that a family of curves $\phi=$ const. is a family of parallels provided the magnitude of $\nabla\phi$ is a function of ϕ only†. Thus if the parametric curves are orthogonal, with $\mathbf{a}$ and $\mathbf{b}$ as unit tangents, the family $u=$ const. will be parallels provided the magnitude of $\mathbf{a}/\sqrt{E}$ is independent of v. This requires $E_2=0$ and therefore $\operatorname{div}\mathbf{b}=0$. But $\mathbf{b}$ is the unit tangent to the curves of the family. Consequently, since the parametric curves may be chosen at pleasure, we have the theorem:

* See also Art. 2 of a paper by the author in *The Mathematical Gazette*, Vol. 13 (1926), pp. 1–6.

† Vol. I, Art. 114.

A necessary and sufficient condition that a family of curves with unit tangent $\mathbf{t}$ *be a family of parallels is that* $\operatorname{div} \mathbf{t}$ *vanish identically.*

Thus the characteristic property of a family of parallels is the vanishing of its divergence. Now $\operatorname{div} \mathbf{t}$ is equal in magnitude to the geodesic curvature of the orthogonal trajectory of the family; and, if this is everywhere zero, these trajectories are geodesics. Hence:

The orthogonal trajectories of a family of parallels constitute a family of geodesics. And, conversely, the orthogonal trajectories of a family of geodesics constitute a family of parallels.

Let $\mathbf{a}$ and $\mathbf{b}$ be the unit tangents to a family of geodesics and to their orthogonal trajectories respectively. Then $\operatorname{div} \mathbf{b}$ vanishes identically; and $\operatorname{div} \mathbf{a} = 0$ is the equation of the line of striction of the family of geodesics. Consider another family of curves cutting the geodesics at a variable angle θ. The unit tangent $\mathbf{t}$ to a curve of this family is then

$$\mathbf{t} = \mathbf{a} \cos \theta + \mathbf{b} \sin \theta,$$

and the unit surface vector $\mathbf{t}'$ perpendicular to $\mathbf{t}$ is

$$\mathbf{t}' = \mathbf{b} \cos \theta - \mathbf{a} \sin \theta.$$

The geodesic curvature of a curve of the family $\mathbf{t}$ has the value

$$\begin{aligned} -\operatorname{div} \mathbf{t}' &= \operatorname{div} (\mathbf{a} \sin \theta - \mathbf{b} \cos \theta) \\ &= \sin \theta \operatorname{div} \mathbf{a} + \mathbf{t} \cdot \nabla \theta \qquad \text{.................(9).} \end{aligned}$$

If the first member of this equation vanishes for all points, the curve is a geodesic. If $\operatorname{div} \mathbf{a}$ is zero the point is on the line of striction of the family of geodesics; and, if the last term vanishes, the curve cuts the geodesics at a constant angle. Hence the theorem*:

If a curve is drawn on a surface so as to cut a family of geodesics, then, provided it has two of the following properties, it will also have the third: (*a*) *that it is a geodesic,* (*b*) *that it is the line of striction of the family of geodesics,* (*c*) *that it cuts the family at a constant angle.*

This is a generalisation of Bonnet's theorem on the generators of a ruled surface. The equation (9) also shows that:

If a family of curves on a surface cut a family of geodesics at an

* *The Mathematical Gazette*, loc. cit., Art. 3.

*angle which is constant along each curve, the line of normal curvature of the former is identical with the line of striction of the latter**.

We may here also prove the following theorem due to Voss, though it is not connected with families of parallels or geodesics.

If on a given surface there are two families of curves cutting each other at a constant angle, and such that, for each family, all the curves have the same constant geodesic curvature, then the surface has constant negative second curvature, or else is a developable.

Let $\mathbf{a}$ and $\mathbf{t}$ be the unit tangents to the two families, the latter being inclined at a constant angle θ to the former. Then, if $\mathbf{b}$ has the usual significance,

$$\mathbf{t} = \mathbf{a}\cos\theta + \mathbf{b}\sin\theta.$$

Since the geodesic curvatures of the two families are constant, $\operatorname{div}\mathbf{b}$ is constant and so also is $\operatorname{div}(\mathbf{a}\sin\theta - \mathbf{b}\cos\theta)$, which has the value $\sin\theta\operatorname{div}\mathbf{a} - \cos\theta\operatorname{div}\mathbf{b}$. Consequently $\operatorname{div}\mathbf{a}$ also is constant. But, in virtue of the Gauss characteristic equation, the second curvature of the surface is given by†

$$\begin{aligned} K &= -\operatorname{div}(\mathbf{a}\operatorname{div}\mathbf{a} + \mathbf{b}\operatorname{div}\mathbf{b}) \\ &= -(\operatorname{div}\mathbf{a})^2 - (\operatorname{div}\mathbf{b})^2, \end{aligned}$$

and is therefore constant and negative. If $\operatorname{div}\mathbf{a}$ and $\operatorname{div}\mathbf{b}$ both vanish identically, K is zero, and the surface is developable.

13. Orthogonal systems of curves. Dupin's theorem, that the sum of the normal curvatures of a surface in any two perpendicular directions at a point is equal to the first curvature of the surface, is only one aspect of a more comprehensive theorem dealing with the curvature of orthogonal systems of curves drawn on the surface. Let the orthogonal system considered be taken as parametric curves, with $\mathbf{a}$ and $\mathbf{b}$ as unit tangents. Then the vector curvature of the curve $v = \text{const.}$ is

$$\mathbf{a}\cdot\nabla\mathbf{a} = \kappa_n\,\mathbf{n} - \mathbf{b}\operatorname{div}\mathbf{b},$$

by Art. 6, and that of the curve $u = \text{const.}$ is

$$\mathbf{b}\cdot\nabla\mathbf{b} = \kappa_n'\,\mathbf{n} - \mathbf{a}\operatorname{div}\mathbf{a}.$$

* *Quarterly Journal*, loc. cit., Art. 7.
† Vol. I, p. 247, or Art. 13 below.

The sum of these vector curvatures has a component $J\mathbf{n}$ normal to the surface, and a component

$$\mathbf{h} = -(\mathbf{a} \operatorname{div} \mathbf{a} + \mathbf{b} \operatorname{div} \mathbf{b}) \quad\text{.................(10)}$$

tangential to the surface. Now the divergence of $\mathbf{h}$ is, by (12) of Art. 2,

$$-\tfrac{1}{2} \operatorname{div}\left(\frac{G_1 \mathbf{r}_1 + E_2 \mathbf{r}_2}{EG}\right) = -\frac{1}{2\sqrt{EG}}\left[\frac{\partial}{\partial u}\left(\frac{G_1}{\sqrt{EG}}\right) + \frac{\partial}{\partial v}\left(\frac{E_2}{\sqrt{EG}}\right)\right]$$

$$= -\frac{1}{\sqrt{EG}}\left[\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial}{\partial u}\sqrt{G}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial}{\partial v}\sqrt{E}\right)\right] = K,$$

in virtue of the Gauss characteristic equation. Hence the theorem:

The sum of the vector curvatures of the two curves of an orthogonal system through any point has a normal component, whose magnitude is equal to the first curvature of the surface, and a tangential component whose divergence is equal to the second curvature at that point.*

And, since the divergence of the normal component $J\mathbf{n}$ is equal to $-J^2$, we also have the result:

The divergence of the vector curvature of an orthogonal system on the surface is invariant, and equal to $K - J^2$.

The tangential component $\mathbf{h}$ of the vector curvature of the orthogonal system is not the same for all systems; but its divergence is a point-function for the surface. To examine this tangential component, and see how it changes with the orthogonal system selected, consider a second orthogonal system, inclined to the above at an angle θ which varies from point to point. The unit tangents to the curves of this system are $\mathbf{a}\cos\theta + \mathbf{b}\sin\theta$ and $\mathbf{b}\cos\theta - \mathbf{a}\sin\theta$. The tangential component $\mathbf{h}'$ of the vector curvature of this system is given by

$$\mathbf{h}' = -(\mathbf{a}\cos\theta + \mathbf{b}\sin\theta) \operatorname{div}(\mathbf{a}\cos\theta + \mathbf{b}\sin\theta)$$
$$-(\mathbf{b}\cos\theta - \mathbf{a}\sin\theta) \operatorname{div}(\mathbf{b}\cos\theta - \mathbf{a}\sin\theta),$$

which, on expansion, reduces to

$$\mathbf{h}' = -(\mathbf{a} \operatorname{div} \mathbf{a} + \mathbf{b} \operatorname{div} \mathbf{b}) - \mathbf{a}(\mathbf{b} \cdot \nabla\theta) + \mathbf{b}(\mathbf{a} \cdot \nabla\theta)$$
$$= \mathbf{h} + \mathbf{n} \times \nabla\theta = \mathbf{h} - \operatorname{rot}(\theta\mathbf{n}) \quad\text{.......................(11).}$$

* *The Mathematical Gazette*, loc. cit., Art. 5.

Thus the vector curvature of the second orthogonal system differs from that of the first by the tangential vector

$$\mathbf{n} \times \nabla\theta = -\operatorname{rot}(\theta\mathbf{n}),$$

whose divergence vanishes identically. If the two orthogonal systems cut at a constant angle, $\nabla\theta$ is zero, and the vector curvatures of the two systems are equal. There is an infinitude of such systems, every two of which cut at a constant angle, and the vector curvature is the same for all these systems. Thus we have the theorem:

The vector curvature is the same for all orthogonal systems that cut each other at a constant angle. If, however, the inclination θ of one system to the other is variable, their curvatures differ by the tangential vector $\operatorname{rot}(\theta\mathbf{n})$, *whose divergence vanishes identically*.*

14. Isometric orthogonal systems†. Consider next an isometric orthogonal system of curves on the surface, and let these be taken as parametric curves. We have seen that a necessary and sufficient condition that the parametric curves may be isometric is‡

$$\frac{\partial^2}{\partial u \partial v} \log \frac{E}{G} = 0 \quad \text{.....................(12).}$$

We may write this

$$\frac{\partial}{\partial u}\left(\frac{E_2}{E}\right) - \frac{\partial}{\partial v}\left(\frac{G_1}{G}\right) = 0,$$

from which it follows that

$$\operatorname{div}\left(\frac{E_2 \mathbf{r}_1}{E\sqrt{EG}} - \frac{G_1 \mathbf{r}_2}{G\sqrt{EG}}\right) = 0,$$

and therefore, if **a** and **b** have their usual meanings,

$$\operatorname{div}(\mathbf{a} \operatorname{div} \mathbf{b} - \mathbf{b} \operatorname{div} \mathbf{a}) = 0 \quad \text{..............(13).}$$

This equation may be transformed in different ways. First it may be expressed

$$\mathbf{a} \cdot \nabla \operatorname{div} \mathbf{b} - \mathbf{b} \cdot \nabla \operatorname{div} \mathbf{a} = 0 \quad \text{..............(14).}$$

* *Ibid.*, Art. 6.

† Arts. 14–16 are taken from a paper by the author "On Isometric Systems of Curves and Surfaces," *American Journal of Mathematics*, Vol. 49 (1927), pp. 527–534.

‡ Vol. I, Art. 39.

Now $-\operatorname{div}\mathbf{b}$ is the geodesic curvature of $v = \text{const.}$, and $\operatorname{div}\mathbf{a}$ is that of $u = \text{const.}$ Also the above analysis is reversible; so that, if (14) is satisfied, the parametric curves form an isometric system. Hence the theorem:

A necessary and sufficient condition that an orthogonal system of curves on a surface may be isometric is that, at each point, the sum of the derivatives of the geodesic curvatures of the two curves, in their own directions, be zero.

Again, the equation (13) may be expressed

$$\operatorname{div}\{(\mathbf{a}\operatorname{div}\mathbf{a} + \mathbf{b}\operatorname{div}\mathbf{b}) \times \mathbf{n}\} = 0,$$

or

$$\operatorname{div}(\mathbf{h} \times \mathbf{n}) = 0,$$

where, as before, $\mathbf{h}$ is the tangential component of the vector curvature of the orthogonal system, defined by

$$\mathbf{h} = -(\mathbf{a}\operatorname{div}\mathbf{a} + \mathbf{b}\operatorname{div}\mathbf{b}).$$

Now the last equation is equivalent to

$$\mathbf{n} \cdot \operatorname{rot}\mathbf{h} = 0 \quad \text{.......................(15).}$$

Since then both $\mathbf{h}$ and $\operatorname{rot}\mathbf{h}$ are tangential to the surface, $\mathbf{h}$ is the gradient of some scalar function ϕ. Thus*

$$\mathbf{h} = \nabla\phi.$$

But we have shown that the divergence of $\mathbf{h}$ is equal to the second curvature of the surface. Consequently

$$K = \operatorname{div}\mathbf{h} = \nabla^2\phi \quad \text{.....................(16).}$$

The above analysis is reversible, so that if (15) is satisfied the parametric curves constitute an isometric system. Hence the theorem:

A necessary and sufficient condition that an orthogonal system of curves on a surface may be isometric is that the tangential component of the vector curvature of the system be the gradient of some scalar point-function, ϕ, on the surface. The second curvature of the surface is then equal to $\nabla^2\phi$.

The value of the function ϕ, when the condition (12) is satisfied, may be found as follows. The vector $\mathbf{h}$ is given by

$$-\mathbf{h} = \frac{G_1\mathbf{r}_1 + E_2\mathbf{r}_2}{2EG}.$$

* Vol. I, Art. 124.

Now, with the usual notation, E and G may be expressed

$$E = \lambda U, \quad G = \lambda V,$$

where U is a function of u only, and V a function of v only. Consequently, on differentiation, we have

$$\frac{E_2}{E} = \frac{G_2}{G} - \frac{V_2}{V}.$$

Thus the above expression for $-\mathbf{h}$ is equivalent to

$$-\mathbf{h} = \tfrac{1}{2}(\nabla \log G - \nabla \log V)$$
$$= \tfrac{1}{2}\nabla \log \lambda.$$

Thus
$$\phi = -\tfrac{1}{2}\log \lambda \qquad\qquad (17),$$

and the second curvature of the surface is given by

$$K = -\tfrac{1}{2}\nabla^2 \log \lambda \qquad\qquad (18).$$

Cor. For any isometric orthogonal system on a developable surface

$$\nabla^2 \log \lambda = 0.$$

15. Oblique trajectories of an isometric system. Consider next an orthogonal system of curves cutting an isometric system at a constant angle. We have seen that the vector curvatures of these two systems are the same. Consequently, since the condition of isometry is satisfied by the one, it is satisfied by the other also; and we have the theorem:

Any orthogonal system of curves on a surface, cutting an isometric system at a constant angle, is also isometric.

A still more general theorem may be proved as follows. We have seen that the vector curvatures of two orthogonal systems, cutting at a variable angle θ, differ by the tangential vector $\nabla\theta \times \mathbf{n}$. If then the condition of isometry is satisfied by one of the systems, it will also be satisfied by the other provided $\nabla\theta \times \mathbf{n}$ is the gradient of some scalar function. This will be the case provided

$$\mathbf{n} \cdot \operatorname{rot}(\nabla\theta \times \mathbf{n}) = 0,$$

that is

$$\mathbf{n} \cdot \{\mathbf{n} \cdot \nabla(\nabla\theta) - \nabla\theta \cdot \nabla\mathbf{n} + (\nabla\theta)\operatorname{div}\mathbf{n} - \mathbf{n}\nabla^2\theta\} = 0.$$

The first three terms vanish identically, showing that the required condition is

$$\nabla^2\theta = 0.$$

Hence the theorem:

An orthogonal system of curves, cutting an isometric orthogonal system at a variable angle θ, will itself be isometric provided

$$\nabla^2\theta = 0.$$

***16. Alternative form of the condition for isometry.** Another form may be found for the condition of isometry of an orthogonal system. By means of (22) of Art. 4, and the derivatives of **a**, **b**, **n** given in Art. 6, it may be verified that

$$\mathbf{b}\cdot\nabla^2\mathbf{a} = -\frac{1}{\sqrt{EG}}\left(JM + \frac{1}{2}\frac{\partial^2}{\partial u\,\partial v}\log\frac{E}{G}\right).$$

Now $M/\sqrt{EG}$ has the value $\mathbf{a}\cdot\operatorname{rot}\mathbf{a}$, being equal to the moment of the family $v =$ const. Hence the above equation shows that the orthogonal system will be isometric provided

$$\mathbf{b}\cdot\nabla^2\mathbf{a} + J\mathbf{a}\cdot\operatorname{rot}\mathbf{a} = 0.$$

Since $\mathbf{b} = \mathbf{n}\times\mathbf{a}$, we may state the theorem†:

A necessary and sufficient condition, that a family of curves with unit tangent **t** *may form with their orthogonal trajectories an isometric system on the surface, is expressed by*

$$(\mathbf{n}\times\mathbf{t})\cdot\nabla^2\mathbf{t} + J\mathbf{t}\cdot\operatorname{rot}\mathbf{t} = 0.$$

Now the quantity $\mathbf{t}\cdot\operatorname{rot}\mathbf{t}$ is zero when the curves are lines of curvature on the surface. Consequently:

If **a**, **b** *are unit vectors in the principal directions, a necessary and sufficient condition that the lines of curvature may form an isometric system of curves is that* $\mathbf{b}\cdot\nabla^2\mathbf{a}$ *vanish identically.*

When the lines of curvature form an isometric system, the surface is sometimes said to be *isothermic*. The equation $\mathbf{b}\cdot\nabla^2\mathbf{a} = 0$ may therefore be regarded as the differential equation satisfied by an isothermic surface.

17. Family of parallel geodesics. Suppose that, on a given surface, there exists a family of curves which are both geodesics and parallels. Let **a** be the unit tangent to the curves and **b** the unit surface vector perpendicular to **a**. Then, since the curves are geodesics, $\operatorname{div}\mathbf{b}$ must vanish identically; and, since they are parallels,

† *Amer. Journ. of Math.*, loc. cit., p. 530.

div **a** is everywhere zero. Now the second curvature of the surface is given by

$$K = -\operatorname{div}(\mathbf{a}\operatorname{div}\mathbf{a} + \mathbf{b}\operatorname{div}\mathbf{b}) \quad \text{............(19).}$$

Hence it vanishes at all points, and the surface is developable. When the surface is developed into a plane, the above curves become parallel straight lines. Conversely, any family of parallel straight lines on the plane corresponds to a family of parallel geodesics on the developable.

Further, $-(\mathbf{a}\operatorname{div}\mathbf{a} + \mathbf{b}\operatorname{div}\mathbf{b})$ is the tangential part of the vector curvature of the family of curves and their orthogonal trajectories; and, since this is zero, the orthogonal system is isometric. Thus:

If a family of curves are both geodesics and parallels on a surface, the surface is developable. Any family of curves cutting the above at a constant angle are also both geodesics and parallels. Any such family and their orthogonal trajectories constitute an isometric system of curves on the surface.

More generally, any orthogonal system cutting one of the above at a variable angle θ will also be isometric provided $\nabla^2\theta = 0$.

18. Flux and circulation*. Consider a curve C drawn on the surface through the points A and B. Let ds be the length of an element of the curve, and **m** the unit surface vector normal to the curve, in an assigned positive sense. Consider also a family of curves on the surface, with unit tangent **t**. We define the *flux* of the family across the curve C from A to B in the sense of **m** by the equation

$$\text{Flux from } A \text{ to } B = \int_A^B \mathbf{t}\cdot\mathbf{m}\,ds.$$

If C is a closed curve, the vector **m** is taken as directed outward from the enclosed region; and the flux across the closed curve is given by the above integral taken right round the curve. If dS is the area of an element of the enclosed region, the Divergence Theorem† gives

$$\iint \operatorname{div}\mathbf{t}\,dS = \int_0 \mathbf{t}\cdot\mathbf{m}\,ds,$$

* *Quarterly Journal*, loc. cit., pp. 359–361.

† Vol. I, Art. 122.

which may be expressed:

The surface integral of the divergence of a family of curves over any region of the surface is equal to the flux of the family across the boundary of the region.

The unit tangent $\mathbf{s}$ to the curve C is the vector $\mathbf{n} \times \mathbf{m}$, the system $\mathbf{m}$, $\mathbf{s}$, $\mathbf{n}$ being right-handed. We define the *circulation* of the family of curves round the closed curve C by the equation

$$\text{Circulation} = \int_o \mathbf{t} \cdot \mathbf{s}\, ds = \int_o \mathbf{t} \cdot d\mathbf{r}.$$

Then, since the geodesic curvature of a curve of the family has the value $\mathbf{n} \cdot \operatorname{rot} \mathbf{t}$, it follows immediately from the Circulation Theorem* that:

The surface integral of the geodesic curvature of a family of curves over any region of the surface is equal to the circulation of the family round the boundary of the region.

Since the divergence of a family of parallels and the geodesic curvature of a family of geodesics vanish identically, it is clear that:

For any closed curve drawn on the surface, the flux of a family of parallels and the circulation of a family of geodesics vanish identically.

Let $\mathbf{a}$ and $\mathbf{b}$ denote as usual the unit tangents to two orthogonal families. Then, in virtue of (19) and the divergence theorem, the total second curvature of the region bounded by the closed curve C has the value

$$\begin{aligned}\iint K\,dS &= -\iint \operatorname{div}(\mathbf{a} \operatorname{div} \mathbf{a} + \mathbf{b} \operatorname{div} \mathbf{b})\, dS \\ &= -\int_o \mathbf{m} \cdot (\mathbf{a} \operatorname{div} \mathbf{a} + \mathbf{b} \operatorname{div} \mathbf{b})\, ds,\end{aligned}$$

provided the families of the orthogonal system present no singularities within the region. If the family $\mathbf{a}$ is one of geodesics, $\operatorname{div} \mathbf{b}$ vanishes identically. And since the geodesics may be chosen at pleasure, subject to possessing no singularities within the region, we have the theorem:

If $\mathbf{t}$ *is the unit tangent to a family of geodesics, the integral*

* Vol. I, Art. 124.

$\int_0 \mathbf{t}\cdot\mathbf{m} \operatorname{div} \mathbf{t} ds$, *taken round a closed curve drawn on the surface, has the same value for all such families, being minus the total second curvature of the region enclosed.*

If however the geodesics of the family are *concurrent at a pole* within the region enclosed by C, the expression for K becomes infinite at this point. We must therefore isolate the pole with (say) a small geodesic circle C', and take the line integral round both curves. Then at C' we have $\mathbf{t} = -\mathbf{m}$, and therefore $\mathbf{t}\cdot\mathbf{m} = -1$. Also, using geodesic polar coordinates*, $E = 1$, $G = H^2$, and

$$\operatorname{div} \mathbf{t} = \operatorname{div} \mathbf{r}_1 = \frac{H_1}{H}.$$

On C' the principal part of $\operatorname{div} \mathbf{t}$ is $1/u$. Hence, as this circle converges to the pole, the limiting value of the line integral round C' is 2π, and we obtain the equation

$$\iint K\, dS = 2\pi - \int_0 \mathbf{t}\cdot\mathbf{m} \operatorname{div} \mathbf{t}\, ds \quad \text{...........(20)}.$$

This formula expresses *the total second curvature of a portion of the surface, with reference to the boundary values of the divergence and the direction of a family of concurrent geodesics, with pole in the region considered*†.

The above theorem is more general than the *Gauss-Bonnet formula*

$$\iint K\, dS = 2\pi - \int_0 \gamma\, ds \text{....................(21)}$$

for the line integral of the geodesic curvature γ of the closed curve C. This formula may be deduced from (20) in the following manner. The closed curve C determines a family of closed parallel curves, converging to a point O within C. The orthogonal trajectories of this family constitute a family of geodesics, with the point O as pole. Apply the above theorem to the curve C and this family of geodesics. Then on the curve C, $\mathbf{t} = \mathbf{m}$, and $\operatorname{div} \mathbf{t}$ is the geodesic curvature γ of the curve. Hence we have the formula (21).

* Vol. I, Art. 57.

† *Quarterly Journal*, loc. cit., p. 361.

EXAMPLES II

1. The following is an alternative proof that the tangential part of rot $\mathbf{t}$ gives the arc-rate of rotation of the tangent plane (Art. 9).

Let $\mathbf{n}$ be the unit normal at P and $\mathbf{n}+\delta\mathbf{n}$ that at a near-by point Q on the curve, at an arc-distance δs from P. Then the vector $\mathbf{n}\times(\mathbf{n}+\delta\mathbf{n})$ is perpendicular to both normals, and its magnitude is, to the first order, the circular measure of the angle between them. Dividing by δs, and taking the limiting value as δs tends to zero, we have for the required arc-rate of rotation

$$\mathbf{n}\times\frac{d\mathbf{n}}{ds}=\mathbf{n}\times(\mathbf{t}\cdot\nabla\mathbf{n})=-\mathbf{n}\times(\mathbf{n}\times\operatorname{rot}\mathbf{t}) \qquad \text{(Art. 8)}$$

$$=\operatorname{rot}\mathbf{t}-(\mathbf{n}\cdot\operatorname{rot}\mathbf{t})\,\mathbf{n},$$

which is the tangential part of rot $\mathbf{t}$.

2. Deduce the same result from (29) of Art. 6, taking the given curve as $v=\text{const.}$

3. With the notation of Art. 12, show that, if $\operatorname{div}\mathbf{b}=0$,

$$\operatorname{div}\mathbf{t}=\cos\theta\operatorname{div}\mathbf{a}+\mathbf{t}'\cdot\nabla\theta,$$

and deduce the theorem*:

If a family of curves on a surface cuts a family of geodesics at an oblique angle which is constant along each orthogonal trajectory of the former, the lines of striction of the two families are identical.

4. Verify the calculation leading to (11) of Art. 13.

5. Show that $\mathbf{c}$ and $\mathbf{d}$ are parallel to conjugate directions on a surface if $(\mathbf{c}\cdot\nabla\mathbf{n})\cdot\mathbf{d}=0$; and hence that the direction of $\mathbf{d}$ is asymptotic provided $(\mathbf{d}\cdot\nabla\mathbf{n})\cdot\mathbf{d}=0$. Deduce the differential equation of the asymptotic lines

$$(d\mathbf{r}\cdot\nabla\mathbf{n})\cdot d\mathbf{r}=0.$$

6. From the relation $\mathbf{n}\cdot\operatorname{rot}\mathbf{t}=0$, satisfied by the unit tangent to a family of geodesics, show that $\mathbf{t}$ is the gradient of some scalar function ψ, and that the curves $\psi=\text{const.}$ are parallels orthogonal to the geodesics, ψ measuring actual geodesic distance from a fixed parallel.

7. On the ruled surface generated by the principal normals of a twisted curve, the moment of the family of generators at a point of the curve is equal to minus the torsion of the curve.

8. The argument of Art. 11, leading to the equation of the *line of striction*, may be amplified as follows. Let $\mathbf{t}$ be the unit tangent to the curve at P, and $\mathbf{t}+\delta\mathbf{t}$ that at an adjacent point Q (Fig. 1, Art. 10) at a distance δs from P along the orthogonal trajectory, in the sense of $\mathbf{b}$. In place of a common perpendicular to adjacent generators of a skew surface, we consider points at

* *Quarterly Journal*, loc. cit., p. 359.

which the projection of $\mathbf{t}+\delta\mathbf{t}$ on $\mathbf{b}$ is a small quantity of the second or higher order. Then, since $\mathbf{t}$ is perpendicular to $\mathbf{b}$, the quantity $\mathbf{b}\cdot\delta\mathbf{t}$ is of the second or higher order. Dividing by δs, and letting δs tend to zero, we have the equation $\mathbf{b}\cdot(\mathbf{b}\cdot\nabla\mathbf{t})=0$ as in Art. 11.

9. Show that, for the family of curves $\phi=\text{const.}$, the unit surface vector normal to the curve is $\psi\nabla\phi$, where $\psi^2=1/(\nabla\phi)^2$, and the unit tangent is $\psi\nabla\phi\times\mathbf{n}$. Hence prove that the divergence of the family is $\mathbf{n}\cdot(\nabla\psi\times\nabla\phi)$, and deduce the equation of the line of striction, and the condition that the curves $\phi=\text{const.}$ may be a family of parallels.

10. With the notation of Ex. 9, show that the geodesic curvature of a curve $\phi=\text{const.}$ has the value $-(\psi\nabla^2\phi+\nabla\psi\cdot\nabla\phi)$, and the moment of the family the value $-\psi^2\nabla\phi\cdot(\text{rot}\,\nabla\phi)$.

11. Show that the tangential component of the vector $\text{rot}\,(\psi\nabla\phi\times\mathbf{n})$ is $-\psi\,(\nabla\phi\cdot\nabla\mathbf{n}+J\nabla\phi)$.

12. If $\mathbf{b}=\psi\nabla\phi$, show that the component of the vector

$$\text{rot}\,(\mathbf{b}\,\text{div}\,\mathbf{b}+\mathbf{b}\times\text{rot}\,\mathbf{b})$$

normal to the surface is $\nabla\,(\psi^2\nabla^2\phi)\times\nabla\phi$.

13. With the notation of Ex. 9, show that the divergence of the family of orthogonal trajectories of $\phi=\text{const.}$ is equal to $\psi\nabla^2\phi+\nabla\phi\cdot\nabla\psi$, and that the geodesic curvature of an orthogonal trajectory has the value $\mathbf{n}\cdot(\nabla\psi\times\nabla\phi)$.

14. If each curve of one family of an isometric system has constant geodesic curvature, so also has each curve of the other family.

15. Show that the orthogonal system which is inclined at an angle θ to the orthogonal system $\mathbf{a}$, $\mathbf{b}$ will be isometric provided

$$\nabla^2\theta+\mathbf{a}\cdot\nabla\gamma+\mathbf{b}\cdot\nabla\gamma'=0,$$

where $\gamma'=\text{div}\,\mathbf{a}$ and $\gamma=-\text{div}\,\mathbf{b}$.

CHAPTER III

FAMILIES OF CURVES (*continued*). OBLIQUE TRAJECTORIES

19. Two differential invariants of a scalar function*. Before proceeding further with the discussion of a family of curves on a surface, we shall find it convenient to mention two other differential invariants of a point-function ϕ, which may be regarded as modifications of $\nabla\phi$. If the lines of curvature are taken as parametric curves we have

$$\nabla\phi = \frac{1}{E}\mathbf{r}_1\phi_1 + \frac{1}{G}\mathbf{r}_2\phi_2, \qquad \nabla = \frac{1}{E}\mathbf{r}_1\frac{\partial}{\partial u} + \frac{1}{G}\mathbf{r}_2\frac{\partial}{\partial v},$$

and consequently

$$\nabla\phi\cdot\nabla = \frac{1}{E}\phi_1\frac{\partial}{\partial u} + \frac{1}{G}\phi_2\frac{\partial}{\partial v}.$$

Applying this operator to the unit normal $\mathbf{n}$ we have

$$\nabla\phi\cdot\nabla\mathbf{n} = -\frac{1}{E}\phi_1\frac{L}{E}\mathbf{r}_1 - \frac{1}{G}\phi_2\frac{N}{G}\mathbf{r}_2.$$

Thus the vector $-\nabla\phi\cdot\nabla\mathbf{n}$ is obtainable from $\nabla\phi$ by multiplying its resolved parts in the principal directions by the corresponding principal curvatures. This function of ϕ is of frequent occurrence and, for the sake of brevity, will be denoted by $\overline{\nabla}\phi$. Thus

$$\overline{\nabla}\phi = -\nabla\phi\cdot\nabla\mathbf{n} \quad\text{......................(1).}$$

It is clearly a differential invariant of ϕ.

Again, if J is the first curvature of the surface,

$$\begin{aligned} J\nabla\phi - \overline{\nabla}\phi &= \left(\frac{L}{E}+\frac{N}{G}\right)\nabla\phi - \left(\frac{\mathbf{r}_1}{E}\frac{L}{E}\phi_1 + \frac{\mathbf{r}_2}{G}\frac{N}{G}\phi_2\right) \\ &= \frac{N}{G}\frac{\mathbf{r}_1}{E}\phi_1 + \frac{L}{E}\frac{\mathbf{r}_2}{G}\phi_2 \\ &= K\left(\frac{E}{L}\frac{\mathbf{r}_1}{E}\phi_1 + \frac{G}{N}\frac{\mathbf{r}_2}{G}\phi_2\right), \end{aligned}$$

* See also Arts. 4 and 5 of a paper by the author "On small Deformations of Surfaces, etc.," *Quarterly Journal*, Vol. 50 (1927), pp. 272–296.

where K as usual denotes the product of the principal curvatures. The vector, which is multiplied by K in the last equation, is obtainable from $\nabla\phi$ by dividing its resolved parts in the principal directions by the corresponding principal curvatures. We shall denote it by $\nabla^*\phi$. Then the last equation may be expressed

$$K\nabla^*\phi = J\nabla\phi - \overline{\nabla}\phi \quad \text{.....................(2).}$$

The same reasoning as above shows that, if $\mathbf{d}$ is any surface vector, $-\mathbf{d}\cdot\nabla\mathbf{n}$ is the vector obtainable from $\mathbf{d}$ by multiplying its resolved parts in the principal directions by the corresponding principal curvatures. And we may also remark that, by taking the gradient of both members of the identity $\mathbf{n}\cdot\nabla\phi = 0$, we find by (19) of Art. 3 that

$$\overline{\nabla}\phi = \mathbf{n}\times \operatorname{rot}\nabla\phi \quad \text{........................(3).}$$

†**20. Related invariants of a vector function.** From the above definitions of $\overline{\nabla}\phi$ and $\nabla^*\phi$ it follows that, if the lines of curvature are taken as parametric curves, the operators $\overline{\nabla}$ and ∇^* may be expressed

$$\overline{\nabla} = \frac{L}{E^2}\mathbf{r}_1\frac{\partial}{\partial u} + \frac{N}{G^2}\mathbf{r}_2\frac{\partial}{\partial v},$$

and

$$\nabla^* = \frac{1}{L}\mathbf{r}_1\frac{\partial}{\partial u} + \frac{1}{N}\mathbf{r}_2\frac{\partial}{\partial v}.$$

With these we may form differential invariants of a vector function $\mathbf{u}$, analogous to the divergence and rotation. These are defined by

$$\overline{\nabla}\cdot\mathbf{u} = \frac{L}{E^2}\mathbf{r}_1\cdot\mathbf{u}_1 + \frac{N}{G^2}\mathbf{r}_2\cdot\mathbf{u}_2 \quad \text{..................(4),}$$

and

$$\overline{\nabla}\times\mathbf{u} = \frac{L}{E^2}\mathbf{r}_1\times\mathbf{u}_1 + \frac{N}{G^2}\mathbf{r}_2\times\mathbf{u}_2 \quad \text{...........(5),}$$

and similar expressions for $\nabla^*\cdot\mathbf{u}$ and $\nabla^*\times\mathbf{u}$.

Forming these invariants of the position vector $\mathbf{r}$ of the current point on the surface we have

$$\overline{\nabla}\cdot\mathbf{r} = \frac{L}{E} + \frac{N}{G} = J,$$

and

$$\nabla^*\cdot\mathbf{r} = \frac{E}{L} + \frac{G}{N} = \frac{J}{K},$$

† The results of Art. 20 will not be required before Art. 105.

so that $$K = (\overline{\nabla} \cdot \mathbf{r}) / \nabla^* \cdot \mathbf{r}.$$

In like manner, applying the operators to the unit normal $\mathbf{n}$, we find

$$\overline{\nabla} \cdot \mathbf{n} = -\left(\frac{L^2}{E^2} + \frac{N^2}{G^2}\right) = 2K - J^2,$$

from which it follows that

$$2K = \overline{\nabla} \cdot \mathbf{n} + (\nabla \cdot \mathbf{n})^2.$$

Similarly it is clear that

$$\nabla^* \cdot \mathbf{n} = \frac{1}{L} \mathbf{r}_1 \cdot \mathbf{n}_1 + \frac{1}{N} \mathbf{r}_2 \cdot \mathbf{n}_2 = -2.$$

These results are all in agreement with the identical relation

$$K\nabla^* = J\nabla - \overline{\nabla} \quad \text{..........................(6)}.$$

Family of Curves ($\phi = \text{const.}$)†

21. Distance function. Line of striction. Moment. Let ϕ be a point-function for a given surface S, and $\phi = \text{const.}$ a singly infinite family of curves on the surface. We propose to show how the fundamental properties of the family may be neatly expressed in terms of differential invariants of ϕ and the *distance function* ψ for the family of curves. This latter function we define as follows. Let $\mathbf{b}$ be the unit surface vector perpendicular to the curve $\phi = \text{const.}$ in the direction of ϕ increasing, that is to say, in the direction of $\nabla\phi$. If then we write

$$\mathbf{b} = \psi\nabla\phi \quad \text{..........................(7)},$$

it is clear that ψ is the reciprocal of the magnitude of $\nabla\phi$, so that

$$\psi^2 = \frac{1}{(\nabla\phi)^2} \quad \text{..........................(8)}.$$

From the definition of $\nabla\phi$ it then follows that the distance in the direction of $\mathbf{b}$ between the adjacent curves ϕ and $\phi + \delta\phi$ is $\psi\delta\phi$ to the first order. Hence the appropriateness of the name "distance function." The curves $\psi = \text{const.}$ on S may be called the *lines of equidistance* for the family $\phi = \text{const.}$

† The substance of Arts. 21–23 was given by the author in a paper "On Families of Curves on a Surface," *Tôhoku Mathematical Journal*, Vol. 30 (1929), pp. 301–306.

The unit tangent $\mathbf{a}$ to the curve $\phi=$ const. at any point is given by

$$\mathbf{a}=\mathbf{b}\times\mathbf{n}=\psi\nabla\phi\times\mathbf{n}\quad\text{.....................(9)},$$

$\mathbf{a}$, $\mathbf{b}$, $\mathbf{n}$ forming a right-handed system. The *divergence* of the family $\phi=$ const. is div $\mathbf{a}$, and therefore has the value

$$\operatorname{div}(\mathbf{b}\times\mathbf{n})=\mathbf{n}\cdot\operatorname{rot}\mathbf{b}=\mathbf{n}\cdot\operatorname{rot}(\psi\nabla\phi)=\mathbf{n}\cdot(\nabla\psi\times\nabla\phi)\text{...(10)},$$

by (16) of Art. 3, since rot $\nabla\phi$ is perpendicular to $\mathbf{n}$. The *line of striction* of the family is the locus of points at which the divergence is zero. Its equation is therefore

$$\mathbf{n}\cdot(\nabla\psi\times\nabla\phi)=0\quad\text{.....................(11)}.$$

The family $\phi=$ const. will be a family of *parallels* if this equation is satisfied identically. This condition is that $\nabla\psi$ and $\nabla\phi$ be parallel, so that ψ must be constant along each curve, the family $\phi=$ const. thus coinciding with its lines of equidistance. The equation (11) may be otherwise expressed

$$\mathbf{a}\cdot\nabla\psi=0.$$

Since the direction of $\mathbf{a}$ is taken as the positive direction along the curve, the *geodesic curvature* of a curve $\phi=$ const. is given by

$$\gamma=-\operatorname{div}\mathbf{b}=-\operatorname{div}(\psi\nabla\phi)=-(\psi\nabla^2\phi+\nabla\psi\cdot\nabla\phi)\text{ ...(12)}.$$

The *line of normal curvature* of the family is the locus of points at which this vanishes. Its equation is therefore

$$\psi\nabla^2\phi+\nabla\psi\cdot\nabla\phi=0\quad\text{..................(13)}.$$

The curves will constitute a family of *geodesics* provided (13) is satisfied identically.

The *moment* of the family $\phi=$ const. at any point has the value $\mathbf{a}\cdot\operatorname{rot}\mathbf{a}$. This is equal to the torsion τ of the geodesic tangent, and is therefore also given by $-\mathbf{b}\cdot\operatorname{rot}\mathbf{b}$, so that

$$\begin{aligned}\tau&=-\psi\nabla\phi\cdot\operatorname{rot}(\psi\nabla\phi)\\&=-\psi\nabla\phi\cdot(\nabla\psi\times\nabla\phi+\psi\operatorname{rot}\nabla\phi)\\&=-\psi^2\nabla\phi\cdot\operatorname{rot}\nabla\phi\quad\text{.............................(14)},\end{aligned}$$

the scalar triple product with a repeated factor being zero. In virtue of (3) this may also be expressed

$$\tau=\psi^2\nabla\phi\cdot(\mathbf{n}\times\overline{\nabla}\phi)=\psi^2\overline{\nabla}\phi\cdot(\nabla\phi\times\mathbf{n})\quad\text{......(15)}.$$

The *line of zero moment* is therefore given by

$$\nabla\phi \cdot \operatorname{rot} \nabla\phi = 0 \quad \text{.....................(16)},$$

or

$$(\mathbf{n} \times \nabla\phi) \cdot \overline{\nabla}\phi = 0,$$

and *the curves* $\phi = $ *const. will be a family of lines of curvature on* S *provided* (16) *is satisfied identically.*

22. Normal curvature. Conjugate direction. The normal curvature of the surface in the direction of $\mathbf{b}$ is equal to $-(\mathbf{b} \cdot \nabla\mathbf{n}) \cdot \mathbf{b}$; and consequently that in the direction of the curve $\phi = $ const. is given by

$$\begin{aligned} \kappa_n &= J + (\mathbf{b} \cdot \nabla\mathbf{n}) \cdot \mathbf{b} \\ &= J + \psi^2 (\nabla\phi \cdot \nabla\mathbf{n}) \cdot \nabla\phi \\ &= J - \psi^2 \overline{\nabla}\phi \cdot \nabla\phi \quad \text{.......................(17)}. \end{aligned}$$

This may also be written

$$\begin{aligned} \kappa_n &= \psi^2 \nabla\phi \cdot (J \nabla\phi - \overline{\nabla}\phi) \\ &= K \psi^2 \nabla\phi \cdot \nabla^*\phi \quad \text{.......................(18)}, \end{aligned}$$

in virtue of (2). The *line of tangential curvature* of the family is the locus of points at which κ_n is zero. Its equation is therefore

$$\nabla\phi \cdot \nabla^*\phi = 0 \quad \text{.......................(19)}.$$

The curves $\phi = $ const. will be a family of *asymptotic lines* provided (19) is satisfied identically.

The *conjugate direction* to that of the curve is the direction of the component of rot $\mathbf{a}$ tangential to the surface; and the magnitude of this component is the arc-rate of rotation of the tangent plane as the point of contact moves along the curve. Now

$$\begin{aligned} \operatorname{rot} \mathbf{a} &= \operatorname{rot} (\psi \nabla\phi \times \mathbf{n}) \\ &= \nabla\psi \times (\nabla\phi \times \mathbf{n}) - \psi (\nabla\phi \cdot \nabla\mathbf{n} + J \nabla\phi + \mathbf{n} \nabla^2\phi), \end{aligned}$$

and the tangential part of this vector is

$$-\psi (\nabla\phi \cdot \nabla\mathbf{n} + J \nabla\phi) = -K \psi \nabla^*\phi.$$

Thus *the direction conjugate to that of the curve* $\phi = $ *const. is the direction of the vector* $\nabla^*\phi$; *and the arc-rate of rotation of the tangent plane along the curve is the magnitude of the vector*

$$-K\psi \nabla^*\phi.$$

The directions of the curves $\phi = $ const. and $\varpi = $ const. will be conjugate if $\nabla\varpi$ is perpendicular to $\nabla^*\phi$. Hence:

The two families of curves $\phi = const.$ and $\varpi = const.$ will form a conjugate system provided

$$\nabla\varpi \cdot \nabla^*\phi = 0 \quad\text{.......................(20)}.$$

This condition may be equally well expressed

$$\nabla\phi \cdot \nabla^*\varpi = 0.$$

Since an asymptotic direction is self-conjugate, the condition (19) that the curves $\phi = \text{const.}$ may be asymptotic lines follows immediately from the above theorem.

The direction conjugate to that of $\phi = \text{const.}$ may be otherwise indicated. For the angle θ of rotation from the direction of **a** to the conjugate direction is given by $\tan\theta = -\kappa_n/\tau$. Hence we may state the theorem:

The family of curves conjugate to the family $\phi = const.$ cuts the latter at a variable angle θ given by

$$\tan\theta = \frac{K\nabla\phi \cdot \nabla^*\phi}{\nabla\phi \cdot \operatorname{rot}\nabla\phi}.$$

We have already seen that the curves $\phi = \text{const.}$ will constitute one family of an *isometric system* provided $\nabla^2\phi/(\nabla\phi)^2$ is a function of ϕ only. This means that the gradient of the first function must be parallel to $\nabla\phi$ at all points of the surface. Consequently, since the vector product of two surface vectors is normal to the surface, we may state the result:

The curves $\phi = const.$ and their orthogonal trajectories will constitute an isometric system provided

$$[\mathbf{n}, \nabla\phi, \nabla(\psi^2\nabla^2\phi)] = 0 \quad\text{..................(21)}.$$

An alternative form for this condition may be found as follows. If we calculate the vector $\operatorname{rot}(\mathbf{b}\operatorname{div}\mathbf{b} + \mathbf{b}\times\operatorname{rot}\mathbf{b})$ from the value of **b** given in (7), we find that its normal component is $\nabla(\psi^2\nabla^2\phi)\times\nabla\phi$. Therefore, since either family of an isometric system may be chosen for the purpose, we have the theorem:

*A family of curves with unit tangent **a** will constitute with their orthogonal trajectories an isometric system on the surface provided*

$$\mathbf{n}\cdot\operatorname{rot}(\mathbf{a}\operatorname{div}\mathbf{a} + \mathbf{a}\times\operatorname{rot}\mathbf{a}) = 0 \quad\text{...........(22)}.$$

Since **a** is a unit vector, this condition may also be expressed

$$\mathbf{n}\cdot\operatorname{rot}(\mathbf{a}\operatorname{div}\mathbf{a} - \mathbf{a}\cdot\nabla\mathbf{a}) = 0.$$

If this relation is satisfied, any other orthogonal system cutting the above at a variable angle θ will also be isometric provided

$$\nabla^2\theta = 0.$$

23. Orthogonal trajectories of the family. The properties of the orthogonal trajectories of the family $\phi = \text{const.}$ may be similarly expressed in terms of ϕ and ψ. The unit tangent to the trajectory is the vector $\mathbf{b}$. The *divergence* of the family of orthogonal trajectories therefore has the value

$$\operatorname{div}(\psi\nabla\phi) = \psi\nabla^2\phi + \nabla\phi\cdot\nabla\psi,$$

being the negative of the geodesic curvature of $\phi = \text{const.}$; and the line of striction of the trajectories is given by (13), being identical with the line of normal curvature of $\phi = \text{const.}$ If the latter be geodesics the former will be parallels.

The *geodesic curvature* of the orthogonal trajectories is $\operatorname{div}\mathbf{a}$, and is equal to the divergence of the family $\phi = \text{const.}$ The *moment* of the family of trajectories is the negative of that of $\phi = \text{const.}$, and is therefore equal to $\psi^2\nabla\phi\cdot\operatorname{rot}\nabla\phi$.

The *normal curvature* κ_n' of the surface in the direction of the orthogonal trajectory is given by

$$\begin{aligned}\kappa_n' &= -(\mathbf{b}\cdot\nabla\mathbf{n})\cdot\mathbf{b} = -\psi^2(\nabla\phi\cdot\nabla\mathbf{n})\cdot\nabla\phi\\ &= \psi^2\nabla\phi\cdot\overline{\nabla}\phi \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(23),\end{aligned}$$

which, by (3), is equivalent to

$$\kappa_n' = \psi^2\nabla\phi\cdot(\mathbf{n}\times\operatorname{rot}\nabla\phi) \quad\ldots\ldots\ldots\ldots(23'),$$

and the trajectories will therefore be *asymptotic lines* provided

$$\nabla\phi\cdot\overline{\nabla}\phi = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(24)$$

is satisfied identically.

The *conjugate direction* to that of the orthogonal trajectory is the direction of the tangential part of $\operatorname{rot}\mathbf{b}$. This component is $\psi\operatorname{rot}\nabla\phi$ or $\psi\overline{\nabla}\phi\times\mathbf{n}$, which is perpendicular to $\overline{\nabla}\phi$. Thus:

The orthogonal trajectories of $\phi = \text{const.}$ and those of $\varpi = \text{const.}$ will form a conjugate system provided

$$\nabla\varpi\cdot\overline{\nabla}\phi = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(25).$$

The condition (24), that the direction of $\nabla\phi$ be asymptotic, is a particular case of (25).

Oblique Trajectories*

24. Line of striction. Parallels. Consider next a family of curves on the surface, cutting the curves $\phi = \text{const.}$ at a variable angle θ. We propose to examine the fundamental properties of this family of trajectories, expressing the results in terms of θ, ϕ, ψ and their differential invariants on the surface.

If the positive direction along the trajectory is obtained by a positive rotation θ from the direction of $\mathbf{a}$, the unit tangent $\mathbf{t}$ to the trajectory is given by

$$\mathbf{t} = \mathbf{a}\cos\theta + \mathbf{b}\sin\theta \quad \text{..................(26)},$$

and the unit surface vector $\mathbf{t}'$, perpendicular to $\mathbf{t}$, by

$$\mathbf{t}' = \mathbf{b}\cos\theta - \mathbf{a}\sin\theta,$$

$\mathbf{t}$, $\mathbf{t}'$, $\mathbf{n}$ forming a right-handed system. The *divergence* of the family of oblique trajectories has the value div $\mathbf{t}$, and is therefore given by

$$\begin{aligned} \operatorname{div}\mathbf{t} &= \operatorname{div}(\mathbf{a}\cos\theta + \mathbf{b}\sin\theta) \\ &= \cos\theta \operatorname{div}\mathbf{a} + \sin\theta \operatorname{div}\mathbf{b} + \mathbf{t}'\cdot\nabla\theta \quad \text{.........(27)}. \end{aligned}$$

On substitution of the values of div $\mathbf{a}$ and div $\mathbf{b}$ found in Art. 21 this becomes

$$\operatorname{div}\mathbf{t} = \cos\theta\,[\mathbf{n}, \nabla\psi, \nabla\phi] + \sin\theta\,(\psi\nabla^2\phi + \nabla\psi\cdot\nabla\phi) + \mathbf{t}'\cdot\nabla\theta \quad \text{......(28)}.$$

The *line of striction* of the family of trajectories, being the locus of points at which the divergence is zero, is obtained by equating to zero the second member of (28). The trajectories will constitute a family of parallels provided the divergence vanishes identically.

Suppose that the curves $\phi = \text{const.}$ are a family of parallels. Then div $\mathbf{a} = 0$; and the trajectories will also constitute a family of parallels provided

$$\sin\theta \operatorname{div}\mathbf{b} + \mathbf{t}'\cdot\nabla\theta = 0,$$

which is equivalent to

$$\mathbf{t}'\cdot\nabla\log\tan\frac{\theta}{2} + \operatorname{div}\mathbf{b} = 0 \quad \text{..............(29)}.$$

Consequently, since $-\operatorname{div}\mathbf{b}$ is the geodesic curvature of the family $\phi = \text{const.}$, we have the theorem:

* The substance of Arts. 24–27 was given by the author in a paper "On oblique Trajectories of a Family of Curves on a Surface," *Crelle's Journal*, Bd. 160 (1928), S. 61–66.

A family of curves cutting a family of parallels at a variable angle θ will themselves be parallels provided $\mathbf{t}' \cdot \nabla \log \tan \frac{\theta}{2}$ *is equal to the geodesic curvature of the family of parallels.*

Similarly, if the curves $\phi = \text{const.}$ are geodesics, $\operatorname{div} \mathbf{b} = 0$; and the trajectories will be a family of parallels provided

$$\mathbf{t}' \cdot \nabla \log \tan \left(\frac{\theta}{2} + \frac{\pi}{4}\right) + \operatorname{div} \mathbf{a} = 0.$$

Hence the theorem:

A family of curves cutting a family of geodesics at a variable angle θ will constitute a family of parallels provided $\mathbf{t}' \cdot \nabla \log \tan \left(\frac{\theta}{2} + \frac{\pi}{4}\right)$ *is equal to the negative of the divergence of the family of geodesics.*

If $\theta = \pi/2$ it follows from (27) that the orthogonal trajectories of the geodesics are parallels, as seen in Art. 12.

25. Geodesic curvature. Various theorems. Since the direction of $\mathbf{t}$ is the positive direction along a curve C of the family of trajectories, the geodesic curvature of the curve C is given by

$$\begin{aligned} \gamma &= -\operatorname{div} \mathbf{t}' = \operatorname{div} (\mathbf{a} \sin \theta - \mathbf{b} \cos \theta) \\ &= \sin \theta \operatorname{div} \mathbf{a} - \cos \theta \operatorname{div} \mathbf{b} + \mathbf{t} \cdot \nabla \theta \quad \text{.........(30).} \end{aligned}$$

Using the values of $\operatorname{div} \mathbf{a}$ and $\operatorname{div} \mathbf{b}$ found in Art. 21, we may write this

$$\gamma = \sin \theta (\nabla \psi \times \nabla \phi \cdot \mathbf{n}) - \cos \theta (\psi \nabla^2 \phi + \nabla \psi \cdot \nabla \phi) + \mathbf{t} \cdot \nabla \theta \text{...(31).}$$

Since $-\operatorname{div} \mathbf{b}$ is the geodesic curvature of the curve $\phi = \text{const.}$, and $\operatorname{div} \mathbf{a}$ that of the orthogonal trajectory, the equation (30) agrees with Liouville's formula* for the geodesic curvature of a curve in terms of those of the orthogonal parametric curves. The *line of normal curvature* of the family of trajectories, being the locus of points at which γ is equal to zero, is obtained by equating to zero the second member of (31). The oblique trajectories will constitute a family of geodesics if γ vanishes identically.

Suppose that the curves $\phi = \text{const.}$ are a family of geodesics. Then $\operatorname{div} \mathbf{b} = 0$; and the trajectories will also be a family of geodesics provided

$$\sin \theta \operatorname{div} \mathbf{a} + \mathbf{t} \cdot \nabla \theta = 0 \quad \text{.................(32),}$$

* Vol. I, Art. 54.

which is equivalent to

$$\mathbf{t}\cdot\nabla \log \tan \frac{\theta}{2} + \operatorname{div} \mathbf{a} = 0 \quad \text{.................(33).}$$

Hence the theorem:

A family of curves cutting a given family of geodesics at a variable angle θ will themselves be geodesics provided $\mathbf{t}\cdot\nabla \log \tan \frac{\theta}{2}$ *is equal to the negative of the divergence of the given family of geodesics.*

Suppose that both families are thus geodesics. The line of striction of $\phi = \text{const.}$ is given by $\operatorname{div} \mathbf{a} = 0$, and therefore, in virtue of (32), by $\mathbf{t}\cdot\nabla\theta = 0$. Consequently, since either family of geodesics may be chosen for reference, we have the theorem:

If two families of geodesics cut at a variable angle θ, the line of striction of either is given by the vanishing of the derivative of θ in the direction of the other.

Next suppose that the curves $\phi = \text{const.}$ are parallels. Then $\operatorname{div} \mathbf{a} = 0$; and it follows from (30) that the trajectories will then be geodesics provided

$$\mathbf{t}\cdot\nabla\theta - \cos\theta \operatorname{div} \mathbf{b} = 0 \quad \text{.................(34),}$$

which may also be expressed

$$\mathbf{t}\cdot\nabla \log \tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) = \operatorname{div} \mathbf{b} \quad \text{..............(35).}$$

Thus:

A family of curves cutting a family of parallels at a variable angle θ will constitute a family of geodesics provided

$$\mathbf{t}\cdot\nabla \log \tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)$$

is equal to the negative of the geodesic curvature of the family of parallels.

If $\theta = \pi/2$ it follows from (34) that the orthogonal trajectories of the parallels are geodesics, as already proved in Art. 12. Also, since the orthogonal trajectories of a family of geodesics constitute a family of parallels, and *vice versa*, two of the above theorems may de deduced from the corresponding theorems of Art. 24.

From (30) we may also derive another result which will prove useful. Suppose that the oblique trajectories of $\phi = \text{const.}$ are

geodesics, and that θ is constant along each of them. Then γ and $\mathbf{t} \cdot \nabla\theta$ both vanish identically. If then θ is an oblique angle, it follows from (30) that the vanishing of either div $\mathbf{a}$ or div $\mathbf{b}$ necessitates the vanishing of the other. Thus, if the curves $\phi = \text{const.}$ are either geodesics or parallels, they are both. But we have already shown that, in this case, the surface is developable*. Hence the theorem:

If a family, A, of geodesics cut a family, B, of geodesics or parallels at an angle which is constant along each member of A, the curves B are both geodesics and parallels, and the surface is developable†.

26. Moment of family. Lines of curvature. The moment m of the family of trajectories at any point has the value $\mathbf{t} \cdot \text{rot}\, \mathbf{t}$, and is therefore given by

$$\begin{aligned} m &= (\mathbf{a} \cos\theta + \mathbf{b} \sin\theta) \cdot \text{rot}\, (\mathbf{a} \cos\theta + \mathbf{b} \sin\theta) \\ &= \cos^2\theta\, (\mathbf{a} \cdot \text{rot}\, \mathbf{a}) + \sin^2\theta\, (\mathbf{b} \cdot \text{rot}\, \mathbf{b}) \\ &\qquad + \sin\theta \cos\theta\, (\mathbf{a} \cdot \text{rot}\, \mathbf{b} + \mathbf{b} \cdot \text{rot}\, \mathbf{a}). \end{aligned}$$

Now $\mathbf{a} \cdot \text{rot}\, \mathbf{a}$ is the moment τ of the family $\phi = \text{const.}$, and is given by (14) or (15). It is the negative of $\mathbf{b} \cdot \text{rot}\, \mathbf{b}$. Also $-\mathbf{b} \cdot \text{rot}\, \mathbf{a}$ is the normal curvature κ_n in the direction of $\mathbf{a}$, and $\mathbf{a} \cdot \text{rot}\, \mathbf{b}$ is the normal curvature κ_n' in the direction of $\mathbf{b}$. These are given by (17) and (23). Thus the moment of the family of trajectories has the value

$$\begin{aligned} m &= \tau (\cos^2\theta - \sin^2\theta) + (\kappa_n' - \kappa_n) \sin\theta \cos\theta \\ &= \tau \cos 2\theta + \tfrac{1}{2} (2\kappa_n' - J) \sin 2\theta \\ &= -\psi^2 (\nabla\phi \cdot \text{rot}\, \nabla\phi) \cos 2\theta + \tfrac{1}{2} (2\psi^2 \nabla\phi \cdot \overline{\nabla}\phi - J) \sin 2\theta \end{aligned} \qquad \text{......(36).}$$

The line of zero moment of the family of trajectories is obtained by equating to zero the second member of (36). If the moment vanishes identically the trajectories are lines of curvature. Hence the theorem:

The angle, measured in the positive sense, from the direction of

* Art. 17.

† Given in Art. 1 of a paper by the author "On Levi-Civita's Theory of Parallelism," *Bull. Amer. Math. Soc.*, Vol. 34 (1928), pp. 585–590.

the curve $\phi = const.$ to a principal direction on the surface, is given by

$$\tan 2\theta = \frac{2\psi^2 \nabla\phi \cdot \text{rot}\,(\nabla\phi)}{2\psi^2 \nabla\phi \cdot \overline{\nabla}\phi - J} \quad \ldots\ldots\ldots\ldots\ldots(37).$$

27. Normal curvature. Conjugate direction. The normal curvature of the surface in the direction of the oblique trajectory is equal to $-\mathbf{t}' \cdot \text{rot}\,\mathbf{t}$. This has the value

$$\begin{aligned}(\mathbf{a} \sin\theta - \mathbf{b}\cos\theta) \cdot \text{rot}\,(\mathbf{a}\cos\theta + \mathbf{b}\sin\theta)& \\ = \kappa_n \cos^2\theta + \kappa_n' \sin^2\theta + 2\tau \sin\theta\cos\theta& \\ = K\psi^2 (\nabla\phi \cdot \nabla^*\phi)\cos^2\theta + \psi^2 (\nabla\phi \cdot \overline{\nabla}\phi)\sin^2\theta& \\ - \psi^2 (\nabla\phi \cdot \text{rot}\,\nabla\phi)\sin 2\theta&.\end{aligned}$$

The line of tangential curvature of the family is obtained by equating this expression to zero. The trajectories will be asymptotic lines if the expression vanishes identically. Thus:

The angle θ, measured in the positive sense, from the direction of the curve $\phi = const.$ to an asymptotic direction on the surface is found from the equation

$$(\nabla\phi \cdot \overline{\nabla}\phi)\tan^2\theta - 2\,(\nabla\phi \cdot \text{rot}\,\nabla\phi)\tan\theta + K\,(\nabla\phi \cdot \nabla^*\phi) = 0.$$

The *direction conjugate* to that of the trajectory is given by the tangential component of rot $\mathbf{t}$. Now

$$\text{rot}\,\mathbf{t} = \cos\theta\,\text{rot}\,\mathbf{a} + \sin\theta\,\text{rot}\,\mathbf{b} - \mathbf{t}' \times \nabla\theta.$$

Inserting the values of rot $\mathbf{a}$ and rot $\mathbf{b}$ we find for this tangential component

$$\psi\,(\text{rot}\,\nabla\phi)\sin\theta - \psi K\,(\nabla^*\phi)\cos\theta \quad \ldots\ldots\ldots(38).$$

Hence the result:

The vector (38) *gives both the direction conjugate to that of the trajectory, and the arc-rate of rotation of the tangent plane along the trajectory.*

EXAMPLES III

1. From (27) of Art. 24, deduce the theorems:

If two families of parallels cut at a variable angle θ, the line of normal curvature of either is given by the vanishing of the derivative of θ in the direction perpendicular to the other†.

If a family, A, of parallels cut a family, B, of geodesics or parallels at an angle which is constant along each orthogonal trajectory of A, the curves B are both geodesics and parallels, and the surface is developable.

2. Deduce the value of κ_n' given in (23) from the formula

$$\kappa_n' = \mathbf{a} \cdot \operatorname{rot} \mathbf{b} = \psi (\nabla\phi \times \mathbf{n}) \cdot \operatorname{rot} (\psi \nabla\phi).$$

3. Deduce formula (18) from (23).

4. By expanding the second member of the identity

$$\operatorname{rot} \mathbf{a} = \operatorname{rot} (\psi \nabla\phi \times \mathbf{n}),$$

deduce (18) from the relations $\kappa_n = -\mathbf{b} \cdot \operatorname{rot} \mathbf{a}$, and $\mathbf{b} = \psi \nabla\phi$.

5. Verify the expression (38) for the tangential part of rot $\mathbf{t}$.

6. Prove that, for any choice of parameters u, v on the surface,

$$\bar{\nabla}\phi = H^{-2}[(GJ - N)\phi_1 - (FJ - M)\phi_2]\mathbf{r}_1 + H^{-2}[(EJ - L)\phi_2 - (FJ - M)\phi_1]\mathbf{r}_2,$$

and $$\nabla^*\phi = [(N\phi_1 - M\phi_2)\mathbf{r}_1 + (L\phi_2 - M\phi_1)\mathbf{r}_2](LN - M^2)^{-1},$$

and verify the identity (2), using (1) of Art. 1.

7. Show that $\bar{\nabla} \times \mathbf{r}$, $\nabla^* \times \mathbf{r}$, $\bar{\nabla} \times \mathbf{n}$, $\nabla^* \times \mathbf{n}$ all vanish identically.

8. Verify that the normal part of the vector $\operatorname{rot}(\mathbf{b} \operatorname{div} \mathbf{b} + \mathbf{b} \times \operatorname{rot} \mathbf{b})$ has the value $\nabla(\psi^2 \nabla^2 \phi) \times \nabla\phi$.

† *Crelle's Journal*, loc. cit., Art. 3.

CHAPTER IV

RULED SURFACES. WEINGARTEN SURFACES

Ruled Surfaces

28. Some general observations. In Chapter VII of our earlier volume some of the elementary properties of ruled surfaces were considered. We shall now briefly supplement that discussion, showing, in particular, how by a different choice of coordinates the analytical treatment may be simplified, and other properties made apparent.

First we observe that, since a generator of a ruled surface is straight, its vector curvature is zero. Consequently the resolved parts of this curvature tangential and normal to the surface both vanish. The generators are therefore both geodesics and asymptotic lines on the surface. Conversely:

If a family of lines on a surface are both geodesics and asymptotic lines, they are straight lines, and the surface is ruled.

For, the components of the vector curvature tangential and normal to the surface are then both zero, so that the vector curvature vanishes identically, and the lines are therefore straight. But, though the curvature of a generator is zero, the torsion does not vanish identically, except in the case of a developable surface. When the line is regarded as a geodesic, its principal normal is the normal to the surface, and its torsion is the arc-rate of rotation of the osculating plane as the point moves along the generator. When, however, the generator is considered as an asymptotic line, its principal normal is tangential to the surface and perpendicular to the generator. Its torsion is then the arc-rate of rotation of the tangent plane along the generator. These two torsions are clearly equal, the torsion of an asymptotic line being equal to that of the geodesic tangent. And, since the torsion of an asymptotic line has the value $\pm\sqrt{-K}$, we have the result:

The arc-rate of rotation of the tangent plane as the point of contact moves along a generator is equal to $\pm\sqrt{-K}$.

The upper sign corresponds to a right-handed surface, and the lower to a left-handed.

Further, the *moment* of the family of generators at any point, being equal to the torsion of the geodesic tangent, has the value $\pm\sqrt{-K}$. Also the parameter of distribution of a ray is, from its definition, equal to the reciprocal of the value of the moment of the family of generators at the central point of the ray*. Thus:

The parameter of distribution for any generator is the value of $\pm 1/\sqrt{-K}$ *at the central point (or point of striction) of the generator.*

29. Choice of coordinates. Curvatures. For the analytical treatment there are distinct advantages in choosing the generators and their orthogonal trajectories as parametric curves. Since the generators are geodesics these trajectories are parallels. Let the generators be taken as curves $v = \text{const.}$, with unit tangent $\mathbf{a}$, and their orthogonal trajectories as curves $u = \text{const.}$, with unit tangent $\mathbf{b}$. Then, if u denotes actual distance along a generator, we have $E=1$ and $F=0$. Since the normal curvature in the direction of a generator is zero, it follows that $L=0$; so that the first and second curvatures are given by

$$J=\frac{N}{G},\quad K=\frac{-M^2}{G} \qquad\text{.....................(1).}$$

Thus K is nowhere positive.

The *distance function*, ψ, for the family of generators, $v = \text{const.}$, is the reciprocal of the magnitude of ∇v. Consequently

$$\psi=\sqrt{G}.$$

It will be convenient to express the various magnitudes in terms of ψ rather than G. Let suffixes 1 and 2 denote, as usual, partial differentiations with respect to u and v. Then the Gauss characteristic equation gives

$$M^2=\frac{G_{11}}{2}-\frac{G_1^{\,2}}{4G}=\psi\psi_{11} \qquad\text{.....................(2),}$$

so that

$$K=-\frac{\psi_{11}}{\psi} \qquad\text{.........................(3),}$$

and the Mainardi-Codazzi relations may be expressed

$$\frac{\partial}{\partial v}\left(\frac{M}{\psi}\right)=\frac{\partial}{\partial u}\left(\frac{N}{\psi}\right) \qquad\text{.....................(4),}$$

* Vol. I, Art. 68.

and
$$\frac{\partial}{\partial u}(M\psi) = 0 \quad \text{.......................(5)}.$$

The *line of striction* of the family of generators is given by $\operatorname{div} \mathbf{a} = 0$. But
$$\operatorname{div} \mathbf{a} = \frac{G_1}{2G} = \frac{\psi_1}{\psi}.$$

Hence the equation of the line of striction is $\psi_1 = 0$. This is otherwise obvious; for the line of striction is the locus of points at which ψ has its least value on the generator. It cuts any generator in its central point, or point of striction.

From (5) it follows that $M\psi$ is constant along any generator. Consequently, in virtue of (1), $K\psi^4$ is constant along any generator. We may express this:

On a given generator of a skew surface the second curvature varies inversely as the fourth power of the distance function.

From this we have, in particular, the known property that, on any ray, K has its maximum value at the central point.

On a developable surface $K = 0$, and therefore $M = 0$, so that the generators are lines of curvature. Conversely, if the generators are lines of curvature, the surface is developable. In this case (4) becomes
$$\frac{\partial}{\partial u}(J\psi) = 0.$$

Now, on any ray of a developable surface, ψ is proportional to the distance from the edge of regression. Hence the theorem:

On any generator of a developable surface the first curvature of the surface varies inversely as the distance from the edge of regression.

Again, the arc-rate of increase of $\sqrt{-K}$ along a generator of a skew surface is given by
$$\frac{\partial}{\partial u}\sqrt{-K} = \frac{\partial}{\partial u}\left(\frac{M}{\psi}\right) = M\psi\frac{\partial}{\partial u}\left(\frac{1}{\psi^2}\right) = -2\sqrt{-K}\operatorname{div} \mathbf{a}.$$

Thus:

If the point of contact move with unit speed along a generator, the angular velocity of the tangent plane is $\pm\sqrt{-K}$, *and the angular acceleration has the value* $\mp 2\sqrt{-K}\operatorname{div} \mathbf{a}$.

The angular acceleration vanishes at the line of striction. From the last equation it is also clear that, if K is constant along each generator, either $K=0$ or $\operatorname{div}\mathbf{a}=0$. In the latter case it follows from the theorem on parallel geodesics (Art. 17) that the surface is developable. Hence the theorem:

If the second curvature is constant along each generator of a ruled surface, the surface is developable.

In the next few articles we shall be concerned mainly with properties peculiar to skew surfaces.

30. Explicit expressions. Since the vector $\mathbf{a}$ is constant along a generator, it is a function of v only, and consequently so also is its derivative $\mathbf{a}_2$. But this derivative is given by

$$\mathbf{a}_2 = M\mathbf{n} + \psi_1\mathbf{b} \quad\quad (6).$$

Hence the second member of (6) is constant both in direction and in magnitude. Since its direction is constant, and ψ_1 vanishes at the line of striction, $\mathbf{a}_2$ has the direction of the normal at the central point of the ray; and the angle ϕ of rotation of the tangent plane, as the point of contact moves along the generator from the central point to the point u, is given by

$$\tan\phi = \frac{\psi_1}{M} \quad\quad (7).$$

Let a suffix zero denote that the value of the quantity is taken for the central point of the ray. Then, because the magnitude of $\mathbf{a}_2$ is constant along a generator, it follows from (6) that

$$M^2 + \psi_1^2 = M_0^2.$$

Since M_0 is a function of v only, we may choose this coordinate so as to make M_0 equal to unity. Then, in virtue of (2), we may write the last equation as

$$\psi\psi_{11} + \psi_1^2 = 1 \quad\quad (8),$$

which gives, on integration with respect to u,

$$\psi\psi_1 = u - \alpha \quad\quad (9),$$

and therefore

$$\psi^2 = (u-\alpha)^2 + \beta^2 \quad\quad (10),$$

where α and β are functions of v only.

By means of these relations explicit expressions may be found for the various magnitudes in terms of u, α and β. The equation

of the line of striction is $\psi_1 = 0$. Consequently the *central point* of any ray is given by

$$u = \alpha.$$

The value of du/dv for the line of striction is therefore α'; and this line cuts the generators at an angle θ such that

$$\tan\theta = \psi_0 \frac{dv}{du} = \pm \frac{\beta}{\alpha'} \quad \text{.................(11).}$$

Again, from (9) and (10) we have

$$\frac{\psi_1}{\psi} = \frac{u-\alpha}{(u-\alpha)^2 + \beta^2} = \frac{u-\alpha}{\psi^2} \quad \text{..............(12),}$$

and consequently, in virtue of (8),

$$\frac{M^2}{\psi^2} = \frac{1-\psi_1^2}{\psi^2} = \frac{\psi^2 - (u-\alpha)^2}{\psi^4} = \frac{\beta^2}{\psi^4} \text{...........(13),}$$

so that

$$M\psi = \beta \text{........................(13'),}$$

the sign of β being chosen as that of M, viz. positive for a right-handed surface and negative for a left-handed. The *second curvature*, K, thus has the value

$$K = -\frac{\beta^2}{\psi^4} \quad \text{.......................(14),}$$

and its value at the central point, $u = \alpha$, is therefore

$$K_0 = -\frac{1}{\beta^2}.$$

Consequently $\beta = \pm 1/\sqrt{-K_0}$, and is therefore identical with the parameter of distribution of the ray.

To find the *first curvature* we must calculate N. Now it follows from (4) and (13) that

$$\frac{\partial}{\partial u}\left(\frac{N}{\psi}\right) = \frac{\partial}{\partial v}\left(\frac{M}{\psi}\right) = \frac{(u-\alpha)^2\beta' + 2\alpha'\beta(u-\alpha) - \beta^2\beta'}{\psi^4},$$

and therefore, on integration with respect to u,

$$\frac{N}{\psi} = \mu - \frac{\beta'(u-\alpha) + \alpha'\beta}{(u-\alpha)^2 + \beta^2} \text{ ....(15),}$$

where μ is a function of v only. Consequently the first curvature, J, has the value

$$J = \frac{1}{\psi}\left[\mu - \frac{\beta'(u-\alpha) + \alpha'\beta}{\psi^2}\right] \quad \text{............(16),}$$

ψ being given by (10).

31. Geometrical illustration. Divergence of generators. From (12) and (13) we have, by division,

$$\frac{\psi_1}{M} = \frac{u-\alpha}{\beta},$$

and therefore, in virtue of (7),

$$u - \alpha = \beta \tan \phi \quad \text{.....................(17)},$$

which is the relation* connecting the distance, $u-\alpha$, from the central point of the ray with the inclination, ϕ, of the tangent plane to the central plane of the ray. Consequently

$$\beta^2 \sec^2 \phi = (u-\alpha)^2 + \beta^2 = \psi^2,$$

so that $$\psi \cos \phi = \pm \beta \quad \text{.........................(18)},$$

and $$\psi \sin \phi = \pm (u - \alpha) \quad \text{....................(19)}.$$

From these we have the known property of a skew surface, that the tangent plane makes half a revolution as the point of contact moves from one end of a generator to the other. And (18) shows that:

On any generator $\cos \phi$ *varies inversely as the distance function.*

An instructive geometrical illustration may be given as follows. Take any plane through the generator CX, and choose rectangular

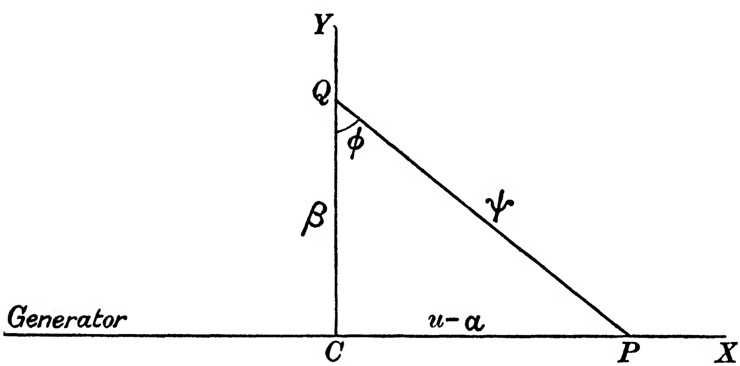

Fig. 2.

axes CX, CY through the central point C so that CX is the direction in which u is measured; and let the sense of rotation from CX to CY be taken as positive. In CY take a point Q so

* Vol. I, Art. 71.

that $CQ = \beta$, CQ having the same sense as CY if β is positive, and the opposite sense if negative. Then, if P is the point of contact of the tangent plane, since $CP = u - \alpha$ it follows from (17) that the angle CQP is equal to ϕ, and the rate of rotation of the tangent plane as P moves along the generator is the rate of rotation of the line QP. In virtue of (10) the length QP is equal to the distance function ψ; and this length is a minimum when P is at the central point. The second curvature of the surface varies inversely as the fourth power of QP, and therefore tends to zero as P becomes infinitely distant.

The *divergence of the family of generators* is given by

$$\operatorname{div} \mathbf{a} = \frac{\psi_1}{\psi} = \frac{u - \alpha}{(u-\alpha)^2 + \beta^2} \quad \text{..............(20).}$$

This vanishes at the central point of the ray, and also tends to zero as u tends to $\pm\infty$. Its stationary values for any ray are given by

$$0 = \frac{\partial}{\partial u} \operatorname{div} \mathbf{a} = \frac{\beta^2 - (u-\alpha)^2}{\psi^4},$$

that is $$u - \alpha = \pm \beta.$$

At these points $\phi = \pm \pi/4$. Consequently:

On any generator the divergence is stationary at the points where the tangent planes are inclined at angles $\pm \pi/4$ to the central plane.

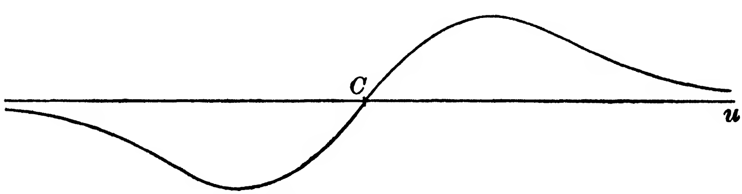

Fig. 3. Divergence of Generators.

The sign of the divergence is that of $u - \alpha$, and therefore changes as P passes through the central point. The graph of $\operatorname{div} \mathbf{a}$, as a function of u, has the shape indicated in the diagram. Its gradient has the value $\cos^2\phi\,(\cos^2\phi - \sin^2\phi)/\beta^2$, being equal to $-K_0$ at the central point.

32. Isometric systems on a skew surface. Consider a family of curves cutting the generators at a variable angle θ.

These are one family of an orthogonal system, whose vector curvature has a tangential part $\mathbf{a}\operatorname{div}\mathbf{a}+\nabla\theta\times\mathbf{n}$ (Art. 13). This orthogonal system will constitute an isometric system of curves on the surface, provided the above vector is the gradient of some scalar function. The condition for this may be expressed

$$\begin{aligned}0&=\mathbf{n}\cdot\operatorname{rot}(\mathbf{a}\operatorname{div}\mathbf{a}+\nabla\theta\times\mathbf{n})\\&=\mathbf{n}\cdot[(\nabla\operatorname{div}\mathbf{a})\times\mathbf{a}+\operatorname{div}\mathbf{a}\operatorname{rot}\mathbf{a}+\mathbf{n}\cdot\nabla(\nabla\theta)\\&\qquad\qquad-\nabla\theta\cdot\nabla\mathbf{n}+\nabla\theta\operatorname{div}\mathbf{n}-\mathbf{n}\nabla^2\theta].\end{aligned}$$

The second term in this expression is zero, since $\mathbf{n}\cdot\operatorname{rot}\mathbf{a}$ is the geodesic curvature of the generator, which vanishes at all points. The next three terms also vanish, since $\mathbf{n}\cdot\nabla\phi\equiv 0$ and $\mathbf{n}$ is perpendicular to its derivatives. Thus the condition reduces to

$$\nabla^2\theta=-\mathbf{b}\cdot\nabla\operatorname{div}\mathbf{a}=-\frac{1}{\psi}\frac{\partial}{\partial v}\left(\frac{\psi_1}{\psi}\right)\ldots\ldots\ldots\ldots(21).$$

Consequently:

A necessary and sufficient condition that a family of curves, cutting the generators at a variable angle θ, may be one family of an isometric orthogonal system on the surface, is that

$$\nabla^2\theta+\frac{1}{\psi}\frac{\partial^2}{\partial u\partial v}\log\psi=0\quad\ldots\ldots\ldots\ldots\ldots(22).$$

In particular, the generators themselves and their orthogonal trajectories will constitute an isometric system provided

$$0=\frac{\partial}{\partial v}\left(\frac{\psi_1}{\psi}\right)=\frac{\alpha'(u-\alpha)^2-2\beta\beta'(u-\alpha)-\alpha'\beta^2}{\psi^4}.$$

This will vanish for all values of u only if

$$\alpha'=\beta'=0.$$

This requires that the parameter of distribution be constant and, in virtue of (11), that the line of striction cut the generators orthogonally.

WEINGARTEN SURFACES

33. Ruled Weingarten surfaces. We have already considered some properties of a W-surface*, chiefly in connection with the two sheets of its evolute. We propose now to continue

* Vol. I, Arts. 76–77.

the study of these surfaces, beginning with Weingarten surfaces which are also ruled surfaces. Let us enquire what conditions must be satisfied in order that a surface may belong to both these classes.

A W-surface is characterised by the property that there exists a functional relation between its first and second curvatures. Hence the lines $J = \text{const.}$ must coincide with the lines $K = \text{const.}$; that is to say, ∇J must be parallel to ∇K, so that

$$J_1 K_2 - J_2 K_1 = 0.$$

Now, with the preceding choice of coordinates for a ruled surface, $K^2 = -M^2/\psi^2$. Hence our condition may be expressed

$$\frac{\partial J}{\partial u}\frac{\partial}{\partial v}\left(\frac{\psi}{M}\right) - \frac{\partial J}{\partial v}\frac{\partial}{\partial u}\left(\frac{\psi}{M}\right) = 0 \quad \ldots\ldots\ldots\ldots\ldots(1).$$

Inserting the values of J and ψ/M given in Art. 30 we see that a necessary and sufficient condition that a ruled surface may also be a W-surface is that the following relation hold identically:

$$\frac{\partial}{\partial u}\left[\frac{\mu}{\psi} - \frac{\beta'(u-\alpha) + \alpha'\beta}{\psi^3}\right]\frac{\partial}{\partial v}\left(\frac{\psi^2}{\beta}\right) - \frac{\partial}{\partial v}\left[\frac{\mu}{\psi} - \frac{\beta'(u-\alpha) + \alpha'\beta}{\psi^3}\right]\frac{2(u-\alpha)}{\beta} = 0.$$

If this is true for all values of u, it holds when $u = \alpha$, in which case it reduces to $\beta' = 0$. Rejecting terms in β' and simplifying the equation we reduce it to

$$\mu'(u-\alpha)^2 + \mu'\beta^2 + \beta\alpha'' = 0.$$

In order that this may hold for all values of u we must have $\mu' = 0$ and $\alpha'' = 0$. Thus the required conditions are that μ, β and α' be constants; and it follows from (11) that the line of striction then cuts the generators at a constant angle. Consequently we have the result, proved by Beltrami and Dini, that:

If a ruled surface is also a W-surface, the parameter of distribution of the generators is constant, and the line of striction cuts the generators at a constant angle.

Bonnet's theorem shows that the line of striction is then a geodesic.

34. Weingarten surfaces in general. The defining property of a W-surface may also be expressed in the form that there exists a functional relation between the principal curvatures. Let the lines of curvature be chosen as parametric curves, and let κ and κ' be the principal curvatures corresponding to the directions of $v=$ const. and $u=$ const. respectively. Then the Mainardi-Codazzi relations are equivalent to

$$\begin{cases} \dfrac{G_1}{G} = \dfrac{2}{\kappa - \kappa'} \dfrac{\partial \kappa}{\partial u} & \text{.......................(2),} \\[2ex] \dfrac{E_2}{E} = \dfrac{-2}{\kappa - \kappa'} \dfrac{\partial \kappa}{\partial v} & \text{.......................(3).} \end{cases}$$

Integrating the first with respect to u, and the second with respect to v, we obtain the formulae

$$\begin{cases} G = Ve^b, \quad b = 2\displaystyle\int \frac{d\kappa'}{\kappa - \kappa'} & \text{..............(4),} \\[2ex] E = Ue^a, \quad a = -2\displaystyle\int \frac{d\kappa}{\kappa - \kappa'} & \text{..............(5),} \end{cases}$$

where U is a function of u only, and V a function of v only. Forming the product of E and G we have the interesting relation

$$EG = \frac{UV}{(\kappa - \kappa')^2} \quad \text{.......................(6).}$$

By taking new parameters u', v' such that $du' = \sqrt{U}du$ and $dv' = \sqrt{V}dv$ we effectively make U and V equal to unity. So we can write, without loss of generality,

$$G = e^b, \quad E = e^a \quad \text{.......................(7),}$$

a and b having the above values. Since then κ' is a function of κ, these equations can be used to express each of the magnitudes E, G in terms of κ alone or κ' alone. And since

$$L = E\kappa, \quad N = G\kappa',$$

the magnitudes L, N may also be expressed in terms of κ alone.

To supplement the theorems concerning the evolute of a W-surface, given in our earlier volume, we may here mention one other. If α, β are the principal radii of curvature of the W-surface, β is a function of α. The square of the linear element of the first sheet

of the evolute is given by*

$$ds^2 = d\alpha^2 + G\left(1 - \frac{\alpha}{\beta}\right)^2 dv^2.$$

Since G is also a function of α, this has the form

$$ds^2 = d\alpha^2 + f(\alpha)\, dv^2.$$

But this form of ds^2 is characteristic of a surface of revolution. The same is true of the linear element of the second sheet of the evolute. Hence we have Weingarten's theorem:

Each sheet of the evolute of a W-surface is applicable to a surface of revolution.

We shall presently consider in some detail the properties of a surface of constant first curvature. Here, however, we may indicate an important result, which follows simply from (4) and (5). If the first curvature is constant we have $\kappa + \kappa' = \text{const.}$, so that $d\kappa' = -d\kappa$. Consequently a and b are equal, and we obtain from (4) and (5) by division

$$\frac{E}{G} = \frac{U}{V}.$$

Thus the condition is satisfied that the orthogonal parametric curves may be isometric, and we have the theorem:

On a surface of constant first curvature the lines of curvature constitute an isometric system.

A surface, whose lines of curvature are isometric, is sometimes said to be *isothermic*. If the lines of curvature are parametric, the surface will be isothermic provided

$$\frac{\partial}{\partial u}\left(\frac{E_2}{E}\right) - \frac{\partial}{\partial v}\left(\frac{G_1}{G}\right) = 0,$$

or, in virtue of (2) and (3), provided

$$\frac{\partial}{\partial u}\left(\frac{1}{\kappa - \kappa'}\frac{\partial \kappa}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\kappa - \kappa'}\frac{\partial \kappa'}{\partial u}\right) = 0,$$

which may also be expressed

$$(\kappa - \kappa')\frac{\partial^2}{\partial u \partial v}(\kappa + \kappa') = \frac{\partial \kappa}{\partial u}\frac{\partial \kappa}{\partial v} - \frac{\partial \kappa'}{\partial u}\frac{\partial \kappa'}{\partial v} \quad \text{.........(8).}$$

In terms of the principal curvatures this expresses a necessary

* Vol. I, Art. 76

and sufficient condition that the surface may be isothermic. It is satisfied, in particular, when κ is a function of one only of the variables u, v, and κ' is also a function of one only. For instance, in the case of a surface of revolution each of the principal curvatures is a function of the parameter determining the parallel orthogonal to the meridians. Consequently, as we have already seen*, a surface of revolution is isothermic. The condition (8) obviously holds also when $\kappa + \kappa'$ is constant.

35. Differential equation satisfied by the unit normal†. In order that a surface may be a W-surface, the unit normal, $\mathbf{n}$, must satisfy a certain differential equation, which may be neatly expressed in terms of the differential invariants of the preceding chapters. Since there exists a functional relation between the first and second curvatures of a W-surface, the gradients of these two functions must be parallel; and, as each of these vectors is tangential to the surface, we have the identical relation

$$(\nabla J \times \nabla K) \cdot \mathbf{n} = 0 \quad \text{.....................(9)}.$$

Conversely, if this condition is satisfied at all points, the surface is a W-surface.

This equation may be expressed in terms of the unit normal. For, as we have already seen,

$$J = -\operatorname{div} \mathbf{n},$$

and

$$2K = \mathbf{n} \cdot \nabla^2 \mathbf{n} + (\operatorname{div} \mathbf{n})^2,$$

and thus, since rot $\mathbf{n}$ is zero, (9) is equivalent to

$$\begin{aligned} 0 &= \nabla J \times [\nabla (\mathbf{n} \cdot \nabla^2 \mathbf{n}) + 2J \nabla J] \cdot \mathbf{n} \\ &= \nabla J \times [(\nabla^2 \mathbf{n}) \cdot \nabla \mathbf{n} + \mathbf{n} \times \operatorname{rot} (\nabla^2 \mathbf{n})] \cdot \mathbf{n}. \end{aligned}$$

Now we have seen that (Art. 4)

$$\nabla^2 \mathbf{n} = (2K - J^2)\, \mathbf{n} - \nabla J,$$

so that

$$(\nabla^2 \mathbf{n}) \cdot \nabla \mathbf{n} = -\nabla J \cdot \nabla \mathbf{n},$$

and the above relation may therefore be written

$$\overline{\nabla} J \cdot (\mathbf{n} \times \nabla J) + \nabla J \cdot \operatorname{rot} (\nabla^2 \mathbf{n}) = 0 \quad \text{...........(10)}.$$

* Vol. I, Art. 39.

† The substance of Arts. 35–38 was given by the author in a paper "On Weingarten Surfaces," *Mathematische Zeitschrift*, Bd. 29, S. 292–299.

Since $J = -\operatorname{div} \mathbf{n}$, this gives one form of the differential equation satisfied by $\mathbf{n}$. The first term of (10) may also be expressed

$$(\mathbf{n} \times \operatorname{rot} \nabla J) \cdot (\mathbf{n} \times \nabla J) \quad \text{or} \quad \nabla J \cdot \operatorname{rot} \nabla J.$$

Hence an alternative form for the condition (10) is

$$(\nabla \operatorname{div} \mathbf{n}) \cdot \operatorname{rot} (\nabla^2 \mathbf{n} - \nabla \operatorname{div} \mathbf{n}) = 0 \qquad \text{...........(11)}.$$

Thus, since the analysis is reversible, we have the theorem:

A necessary and sufficient condition that a given surface may be a W-surface is that the unit normal satisfy the differential equation (10) *or* (11).

36. Lines of constant curvature. Since the curves $J=$const. coincide with the curves $K=$const., these may be called the *lines of constant curvature.* Their fundamental properties may be neatly expressed in terms of differential invariants of $\mathbf{n}$. The family is given by $\operatorname{div} \mathbf{n} = $ const. The distance function, ψ, for the family is defined by

$$\psi^2 = \frac{1}{(\nabla \operatorname{div} \mathbf{n})^2} = \frac{1}{(\nabla J)^2}.$$

The line of striction of the family, being the locus of points at which the divergence is zero, is given by

$$\mathbf{n} \cdot (\nabla \psi \times \nabla J) = 0.$$

The geodesic curvature γ of a line of constant curvature is

$$\gamma = -\operatorname{div} (\psi \nabla J) = -(\psi \nabla^2 J + \nabla \psi \cdot \nabla J).$$

The moment, τ, of the family, or the torsion of the geodesic tangent, is given by

$$\tau = -\psi \nabla J \cdot \operatorname{rot} (\psi \nabla J) = -\psi^2 \nabla J \cdot \operatorname{rot} \nabla J,$$

and the curves will be a family of lines of curvature provided this vanishes identically. It should be observed that the first term in (10) is proportional to τ.

The normal curvature, κ_n, of the surface in the direction of the line of constant curvature has the value expressed by

$$\kappa_n = J - \psi^2 (\nabla J \times \mathbf{n}) \cdot \operatorname{rot} \nabla J.$$

37. Surface of constant first curvature. Consider next the particular class of *W*-surface on which the first curvature, J,

is constant. The *lines of constant curvature* are given by $K = \text{const.}$ Also

$$\mathbf{n} \cdot \nabla^2 \mathbf{n} = 2K - J^2.$$

If then we denote the function $\mathbf{n} \cdot \nabla^2 \mathbf{n}$ by ϕ, the lines of constant curvature are the lines $\phi = \text{const.}$ The fundamental properties of this family of curves are then given by the formulae of Chapter III.

Let the lines of curvature be taken as parametric curves, with unit tangents $\mathbf{a}$ and $\mathbf{b}$; and let the difference of the principal curvatures be denoted by B. Thus

$$B = \kappa - \kappa' = 2\kappa - J \quad \text{..................(12)},$$

and, since J is constant, it follows that

$$\nabla \kappa = -\nabla \kappa' = \tfrac{1}{2} \nabla B \quad \text{.....................(13)}.$$

The lines of constant curvature are the lines along which each of the quantities κ, κ' and K is constant. Now, the Mainardi-Codazzi relations (2) and (3) show that

$$\left.\begin{aligned} \operatorname{div} \mathbf{a} &= \frac{1}{2G\sqrt{E}} \frac{\partial G}{\partial u} = \frac{1}{B\sqrt{E}} \frac{\partial \kappa'}{\partial u} \\ \operatorname{div} \mathbf{b} &= \frac{1}{2E\sqrt{G}} \frac{\partial E}{\partial v} = -\frac{1}{B\sqrt{G}} \frac{\partial \kappa}{\partial v} \end{aligned}\right\} \quad \text{............(14)}.$$

Therefore, in consequence of (13),

$$\begin{aligned} -(\mathbf{a} \operatorname{div} \mathbf{a} + \mathbf{b} \operatorname{div} \mathbf{b}) &= \frac{1}{B}\left(\frac{\mathbf{a}}{\sqrt{E}} \frac{\partial \kappa}{\partial u} + \frac{\mathbf{b}}{\sqrt{G}} \frac{\partial \kappa}{\partial v}\right) \\ &= \frac{1}{B} \nabla \kappa = \tfrac{1}{2} \nabla \log B \quad \text{.........(15)}. \end{aligned}$$

Now the vector in the first member is the tangential part of the vector curvature of the lines of curvature; and (15) shows that this is expressible as the gradient of a scalar function. Consequently, by the theorem of Art. 14, *the lines of curvature constitute an isometric system on a surface of constant first curvature,* as we have already proved in Art. 34. It follows that any orthogonal system cutting the lines of curvature at an angle θ will form an isometric system provided $\nabla^2 \theta = 0$.

Let $\mathbf{h}$ denote the vector in the first member of (15). This equation shows that $\mathbf{h}$ is parallel to $\nabla \kappa$, and therefore perpendicular to the line of constant curvature. Consequently, the tangent

to this curve has the direction of the vector $(\mathbf{b}\,\mathrm{div}\,\mathbf{a} - \mathbf{a}\,\mathrm{div}\,\mathbf{b})$; and, since $-\mathrm{div}\,\mathbf{b}$ and $\mathrm{div}\,\mathbf{a}$ are the geodesic curvatures of the lines of curvature, we have:

The lines $K = const.$ are orthogonal to the resultant vector curvature of the lines of curvature, and are inclined to the principal direction $\mathbf{a}$ *at an angle θ given by*

$$\tan\theta = -\frac{\mathrm{div}\,\mathbf{a}}{\mathrm{div}\,\mathbf{b}} = \frac{\text{geodesic curvature of } u = \text{const.}}{\text{geodesic curvature of } v = \text{const.}}.$$

Other interesting relations between the above quantities may be noticed in passing. Since the second curvature is equal to $\mathrm{div}\,\mathbf{h}$, it follows from (15) that

$$K = \mathrm{div}\,\mathbf{h} = \tfrac{1}{2}\nabla^2 \log B \quad\text{.................(16).}$$

The gradients of κ, κ' and K are all parallel to $\mathbf{h}$. Their values are found as follows. From (15) we have

$$\nabla\kappa = -\nabla\kappa' = B\mathbf{h} \quad\text{....................(17),}$$

and therefore, since $\nabla B = 2\nabla\kappa$,

$$\nabla^2\kappa = \mathrm{div}\,(B\mathbf{h}) = B(2\mathbf{h}^2 + K),$$

and $\nabla^2\kappa'$ is the negative of this. Similarly

$$\nabla K = \nabla(\kappa\kappa') = (\kappa' - \kappa)\nabla\kappa = -B^2\mathbf{h} \quad\text{.........(18),}$$

and consequently

$$\nabla^2 K = -\mathrm{div}\,(B^2\mathbf{h}) = -B^2(4\mathbf{h}^2 + K),$$

the quantity $\mathbf{h}^2$ being the "square" of the tangential curvature of the lines of curvature, having the value $(\mathrm{div}\,\mathbf{a})^2 + (\mathrm{div}\,\mathbf{b})^2$.

38. Surfaces of revolution. A surface of revolution is a W-surface on which the curves $J = \text{const.}$ and the curves $K = \text{const.}$ are one and the same system of lines of curvature. For, if u is the parameter determining the parallel, and v that determining the meridian, J and K are both functions of u only. Thus ∇J is parallel to a line of curvature, so that the derivative of $\mathbf{n}$ in the direction of ∇J is parallel to ∇J. We may express this by the equation

$$(\nabla J \cdot \nabla\mathbf{n}) \times \nabla J \cdot \mathbf{n} = 0 \quad\text{.................(19).}$$

Now the first member of this equation is identical with the first term of (10). Hence the result:

On a surface of revolution the unit normal satisfies the pair of equations

$$\left.\begin{aligned}(\nabla J\cdot\nabla\mathbf{n})\cdot(\mathbf{n}\times\nabla J)=0\\ \nabla J\cdot\operatorname{rot}(\nabla^2\mathbf{n})=0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots(20),$$

which may be expressed in the alternative form

$$\left.\begin{aligned}(\nabla\operatorname{div}\mathbf{n})\cdot\operatorname{rot}(\nabla\operatorname{div}\mathbf{n})=0\\ (\nabla\operatorname{div}\mathbf{n})\cdot\operatorname{rot}(\nabla^2\mathbf{n})=0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots(21).$$

Conversely, if $\mathbf{n}$ satisfies both these conditions, the surface is a W-surface; and, unless ∇J vanishes identically, the lines $J=$ const. and $K=$ const. are one system of lines of curvature. The surface is therefore a surface of revolution. If, however, J is constant, both the above conditions are satisfied in consequence. In order that the lines $K=$ const. may be a system of lines of curvature, it is necessary and sufficient that

$$(\nabla K\cdot\nabla\mathbf{n})\times\nabla K\cdot\mathbf{n}=0\quad\ldots\ldots\ldots\ldots\ldots(22).$$

Now, since J is constant,

$$\nabla^2\mathbf{n}=(2K-J^2)\,\mathbf{n},$$

and therefore $\qquad\operatorname{rot}\nabla^2\mathbf{n}=2\nabla K\times\mathbf{n}.$

In virtue of this relation (22) may be expressed in the alternative form

$$(\mathbf{n}\times\operatorname{rot}\nabla^2\mathbf{n})\cdot\nabla\mathbf{n}\cdot(\operatorname{rot}\nabla^2\mathbf{n})=0\quad\ldots\ldots\ldots\ldots(23).$$

Consequently, a surface of constant first curvature will be a surface of revolution provided the unit normal satisfies the equation (23).

EXAMPLES IV

Ruled Surfaces

1. *If, on a skew surface, the second curvature is constant along each orthogonal trajectory of the generators, then on any one generator the first curvature varies inversely as the distance function, being therefore greatest at the central point.*

2. The moment of a family of oblique trajectories, cutting the generators at an angle θ, has the value

$$\tfrac{1}{2}J\sin 2\theta\pm\sqrt{-K}\cos 2\theta.$$

These trajectories will be lines of curvature if

$$\tan 2\theta=\mp 2\sqrt{-K}/J,$$

the upper sign being taken for a right-handed surface, and the lower for a left-handed.

3. With the notation of Ex. 2, the normal curvature in the direction of the trajectory has the value

$$\sin\theta\,(J\sin\theta \pm 2\sqrt{-K}\cos\theta).$$

Hence the curved asymptotic lines cut the generators at an angle θ given by

$$\tan\theta = \mp 2\sqrt{-K}/J.$$

4. Show that, for a ruled surface,

$$\operatorname{rot}\mathbf{a} = \frac{M}{\psi}\mathbf{a} = \pm\mathbf{a}\sqrt{-K},$$

$$\mathbf{a}\cdot\nabla\mathbf{n} = \mp\sqrt{-K}\,\mathbf{b},$$

$$\mathbf{b}\cdot\nabla\mathbf{n} = \mp\sqrt{-K}\,\mathbf{a} - J\mathbf{b}.$$

5. Prove that, for a skew surface,

$$(\operatorname{div}\mathbf{a})^2 = K + \sqrt{KK_0},$$

$$\frac{\partial}{\partial u}\operatorname{div}\mathbf{a} = -2K - \sqrt{KK_0},$$

and also that

$$\operatorname{div}\mathbf{a} = -\frac{1}{4}\frac{\partial}{\partial u}\log(-K).$$

6. Show that, for a skew surface,

$$\frac{\partial K}{\partial u} = -\frac{4K\psi_1}{\psi}, \quad \frac{\partial K}{\partial v} = -\frac{2M}{\psi}\frac{\partial}{\partial u}(J\psi),$$

and

$$\frac{\partial^2}{\partial u^2}\left(\frac{N}{\psi}\right) = -2\frac{\partial}{\partial v}\left(\frac{M\psi_1}{\psi^2}\right).$$

7. If the point of contact move along a generator with unit speed, prove that the angular acceleration of the tangent plane is stationary when that plane is inclined at an angle $\pm\pi/6$ to the central plane of the ray.

8. Show that

$$\mathbf{a}_{22} = \left(M_2 + \frac{\psi_1}{\psi}N\right)\mathbf{n} - (M^2 + \psi_1^2)\,\mathbf{a} + \left(\psi_{12} - \frac{MN}{\psi}\right)\mathbf{b}.$$

9. Verify that

$$\operatorname{div}\nabla^* K = \frac{4}{\psi}\frac{\partial}{\partial v}\left(\frac{M\psi_1}{\psi^2}\right) - \frac{2}{\psi}\frac{\partial}{\partial u}\left[\frac{1}{\psi^2}\frac{\partial}{\partial u}(N\psi)\right].$$

10. The line of striction of the generators of a skew surface is an orthogonal trajectory of the latter only if these are the binormals of a curve, or if the surface is a right helicoid.

Weingarten Surfaces

11. Show that, for a surface of constant first curvature, with the notation of Art. 34,

$$E = \frac{U}{\kappa - \kappa'}, \quad G = \frac{V}{\kappa - \kappa'},$$

$$L = \frac{\kappa U}{\kappa - \kappa'}, \quad N = \frac{\kappa' V}{\kappa - \kappa'},$$

and that, for a minimal surface,

$$E = U'/\kappa, \qquad G = V'/\kappa',$$
$$L = U', \qquad N = V',$$

where U' is a function of u only, and V' a function of v only.

12. Show that, for a surface of constant second curvature ($\kappa\kappa' = c^2$),

$$E = \frac{U}{\kappa^2 - c^2}, \qquad G = \frac{\kappa^2 V}{\kappa^2 - c^2},$$
$$L = \frac{\kappa U}{\kappa^2 - c^2}, \qquad N = \frac{c^2 \kappa V}{\kappa^2 - c^2},$$

and that, for a developable surface ($\kappa' = 0$),

$$E = U/\kappa^2, \quad G = V, \quad L = U/\kappa, \quad N = 0.$$

13. For a surface of constant second curvature ($\kappa\kappa' = c^2$),

$$\nabla\kappa' = -\frac{c^2}{\kappa^2}\nabla\kappa = -\frac{\kappa'}{\kappa}\nabla\kappa.$$

Also, since
$$\mathbf{b}\cdot\nabla\kappa = -B\,\mathrm{div}\,\mathbf{b},$$
and
$$\mathbf{a}\cdot\nabla\kappa = -\frac{c^2}{\kappa'^2}\mathbf{a}\cdot\nabla\kappa' = -\frac{\kappa}{\kappa'}B\,\mathrm{div}\,\mathbf{a},$$
it follows that
$$\nabla\kappa = -B\left(\frac{\kappa}{\kappa'}\mathbf{a}\,\mathrm{div}\,\mathbf{a} + \mathbf{b}\,\mathrm{div}\,\mathbf{b}\right).$$

The line of constant curvature is perpendicular to this vector, and therefore makes with the direction of $\mathbf{a}$ an angle θ given by

$$\tan\theta = -\frac{\kappa\,\mathrm{div}\,\mathbf{a}}{\kappa'\,\mathrm{div}\,\mathbf{b}}.$$

14. On a surface of constant first curvature

$$\mathrm{rot}\,\nabla^2\mathbf{n} = 2\nabla K \times \mathbf{n}.$$

Hence $\mathrm{rot}\,\nabla^2\mathbf{n}$ is tangential to the line of constant curvature; and the unit tangent $\mathbf{t}$ to this line is

$$\mathbf{t} = \varpi\,\mathrm{rot}\,\nabla^2\mathbf{n},$$

where
$$\varpi^2 = \frac{1}{(\mathrm{rot}\,\nabla^2\mathbf{n})^2}.$$

Hence show that the line of striction of the family of lines of constant curvature is given by

$$\nabla\varpi\cdot\mathrm{rot}\,\nabla^2\mathbf{n} = 0,$$

the geodesic curvature by

$$\gamma = -\mathrm{div}\,(\varpi\mathbf{n}\times\mathrm{rot}\,\nabla^2\mathbf{n}) = \mathbf{n}\cdot\mathrm{rot}\,(\varpi\,\mathrm{rot}\,\nabla^2\mathbf{n}),$$

the moment of the family by

$$\tau = \varpi^2\,(\mathrm{rot}\,\nabla^2\mathbf{n})\cdot(\mathrm{rot}\,\mathrm{rot}\,\nabla^2\mathbf{n}),$$

and the normal curvature in the direction of the curve by

$$\kappa_n = -\varpi^2\,(\mathbf{n}\times\mathrm{rot}\,\nabla^2\mathbf{n})\cdot(\mathrm{rot}\,\mathrm{rot}\,\nabla^2\mathbf{n}).$$

15. Show that, for a surface in which $\kappa - \kappa' = \text{const.}$, the difference of the tangential parts of the vector curvatures of the lines of curvature is the gradient of a scalar function.

16. Prove that the asymptotic lines on a minimal surface are isometric.

17. Show that, for any surface,

$$\nabla J \times \nabla K = (\kappa - \kappa') \nabla\kappa \times \nabla\kappa'.$$

18. Prove that the helicoids are W-surfaces, and are the only W-surfaces on which the lines of curvature cut the lines of constant curvature at a constant angle.

CHAPTER V

CURVILINEAR COORDINATES IN SPACE. DIFFERENTIAL INVARIANTS

39. Notation. Fundamental magnitudes*. Some properties of a triply orthogonal system of surfaces were examined in Chapter XI of our earlier volume, by means of "orthogonal" curvilinear coordinates associated with the system. We shall now consider non-orthogonal systems and "oblique" curvilinear coordinates. By means of the latter we shall also define and discuss the three-parametric gradient, divergence and rotation, as well as other differential invariants of functions in space. These are the invariants which, in recent times, have entered largely into the theory of Mathematical Physics. We shall see that they may also play an important part in Differential Geometry.

A quantity which assumes one or more definite values at each point of a region of space is said to be a function of position, or a point-function, in that region. If it has only one value at each point, the function is said to be uniform or single-valued. We shall be concerned with both scalar and vector point-functions; but all the functions considered will be uniform, and will be assumed free from singularities, unless otherwise stated.

A surface, over which a scalar point-function has a constant value, is called a *level-surface* for that function. Let u, v, w be three point-functions in space, and

$$u = \text{const.}, \quad v = \text{const.}, \quad w = \text{const.} \quad \text{............(1)}$$

the corresponding level-surfaces. The position vector, $\mathbf{r}$, of a point in the space occupied by these surfaces, may be expressed as a function of u, v, w. Thus

$$\mathbf{r} = \mathbf{r}\,(u, v, w).$$

Then u, v, w are regarded as "parameters" fixing the position of the point $\mathbf{r}$, and the surfaces (1) as "parametric surfaces."

* The substance of Arts. 39–41, 43, 45 and 50 was given by the author in a paper "On triple Systems of Surfaces and non-orthogonal Curvilinear Coordinates," *Proc. Royal Soc. Edin.*, Vol. 46 (1926), pp. 194–205.

We shall employ suffixes 1, 2, 3 to denote partial differentiations with respect to u, v, w respectively. Thus

$$\mathbf{r}_1 = \frac{\partial \mathbf{r}}{\partial u}, \quad \phi_2 = \frac{\partial \phi}{\partial v}, \quad \mathbf{r}_{23} = \frac{\partial^2 \mathbf{r}}{\partial v\, \partial w},$$

and so on. Corresponding to the fundamental magnitudes of the first order for a single surface, we have the functions defined by

$$\left.\begin{array}{lll} a = \mathbf{r}_1{}^2, & b = \mathbf{r}_2{}^2, & c = \mathbf{r}_3{}^2 \\ f = \mathbf{r}_2 \cdot \mathbf{r}_3, & g = \mathbf{r}_3 \cdot \mathbf{r}_1, & h = \mathbf{r}_1 \cdot \mathbf{r}_2 \end{array}\right\} \quad \ldots\ldots\ldots\ldots (2).$$

The vector $\mathbf{r}_1$ is tangential to the parametric curve $v = \text{const.}$, $w = \text{const.}$; and the unit vector in this direction is $\mathbf{r}_1/\sqrt{a}$. Similarly

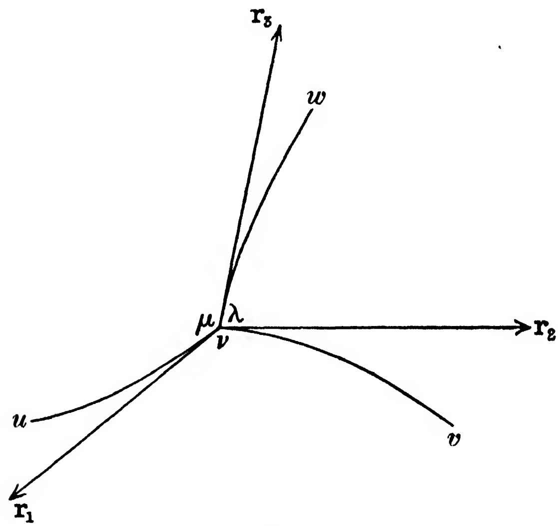

Fig. 4.

the unit vectors tangential to the other parametric curves through the point are $\mathbf{r}_2/\sqrt{b}$ and $\mathbf{r}_3/\sqrt{c}$. The angle λ between the directions of $\mathbf{r}_2$ and $\mathbf{r}_3$ is given by

$$\cos\lambda = \frac{\mathbf{r}_2 \cdot \mathbf{r}_3}{\sqrt{bc}} = \frac{f}{\sqrt{bc}},$$

and similarly the angles μ, ν between the other parametric curves are such that

$$\cos\mu = \frac{g}{\sqrt{ca}}, \quad \cos\nu = \frac{h}{\sqrt{ab}}.$$

The element of arc, ds, of any curve is found from the relation

$$ds^2 = (d\mathbf{r})^2 = (\mathbf{r}_1 du + \mathbf{r}_2 dv + \mathbf{r}_3 dw)^2$$
$$= a\,du^2 + b\,dv^2 + c\,dw^2 + 2f\,dv\,dw + 2g\,dw\,du + 2h\,du\,dv.$$

The volume dV of the element of space bounded by the parametric surfaces u, $u+du$, v, $v+dv$, w, $w+dw$ is the scalar product of the three vectors $\mathbf{r}_1 du$, $\mathbf{r}_2 dv$, $\mathbf{r}_3 dw$. Thus

$$dV = [\mathbf{r}_1 du,\ \mathbf{r}_2 dv,\ \mathbf{r}_3 dw] = [\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3]\, du\,dv\,dw = p\,du\,dv\,dw,$$

in which we have put for brevity

$$p = [\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3].$$

This triple product, denoted by p, will occur frequently in the following pages.

40. Unit normals to the parametric surfaces. The normal to the surface $u = \text{const.}$ is parallel to the vector $\mathbf{r}_2 \times \mathbf{r}_3$. To express this vector as the sum of components in the directions of the parametric curves we have, by the usual formula,

$$p\mathbf{r}_2 \times \mathbf{r}_3 = [\mathbf{r}_2 \times \mathbf{r}_3, \mathbf{r}_2, \mathbf{r}_3]\,\mathbf{r}_1 + [\mathbf{r}_2 \times \mathbf{r}_3, \mathbf{r}_3, \mathbf{r}_1]\,\mathbf{r}_2 + [\mathbf{r}_2 \times \mathbf{r}_3, \mathbf{r}_1, \mathbf{r}_2]\,\mathbf{r}_3.$$

Now the coefficient of $\mathbf{r}_1$ in the second member is equal to

$$(\mathbf{r}_2 \times \mathbf{r}_3)\cdot(\mathbf{r}_2 \times \mathbf{r}_3) = \mathbf{r}_2^2\,\mathbf{r}_3^2 - (\mathbf{r}_2 \cdot \mathbf{r}_3)^2 = bc - f^2.$$

Similarly the coefficient of $\mathbf{r}_2$ is $fg - ch$, and that of $\mathbf{r}_3$ is $hf - bg$. It will be observed that these coefficients are the cofactors of a, h, g respectively in the determinant D defined by

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

Denoting the cofactor of each element by the corresponding capital, we may write the above result

$$p\mathbf{r}_2 \times \mathbf{r}_3 = A\mathbf{r}_1 + H\mathbf{r}_2 + G\mathbf{r}_3 \quad \text{..............(3)},$$

which gives the required resolution. And, on forming the scalar product of each side with $\mathbf{r}_1$, we have

$$p^2 = Aa + Hh + Gg = D \quad \text{.................(4)}.$$

The *unit normal* to the surface $u = \text{const.}$ is then

$$\frac{\mathbf{r}_2 \times \mathbf{r}_3}{\sqrt{bc - f^2}} = \frac{1}{p\sqrt{A}}\,(A\mathbf{r}_1 + H\mathbf{r}_2 + G\mathbf{r}_3) \quad \text{...........(5)}.$$

By the same method as that used in proving (3) we have the corresponding formulae

$$\left.\begin{aligned} p\,\mathbf{r}_3\times\mathbf{r}_1 &= H\mathbf{r}_1+B\mathbf{r}_2+F\mathbf{r}_3 \\ p\,\mathbf{r}_1\times\mathbf{r}_2 &= G\mathbf{r}_1+F\mathbf{r}_2+C\mathbf{r}_3 \end{aligned}\right\} \qquad (6),$$

and

and the unit normals to the surfaces $v=$ const. and $w=$ const. are $\mathbf{r}_3\times\mathbf{r}_1/\sqrt{B}$ and $\mathbf{r}_1\times\mathbf{r}_2/\sqrt{C}$ respectively. The values of the cofactors are given by

$$\left.\begin{aligned} A &= bc-f^2, & B &= ca-g^2, & C &= ab-h^2 \\ F &= gh-af, & G &= hf-bg, & H &= fg-ch \end{aligned}\right\} \qquad (7).$$

It is well known that the determinant

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$$

is equal to D^2; and that the cofactors of the elements of this determinant are proportional to the corresponding elements of the determinant D. Thus

$$\left.\begin{aligned} BC-F^2 &= Da, & CA-G^2 &= Db, & AB-H^2 &= Dc \\ GH-AF &= Df, & HF-BG &= Dg, & FG-CH &= Dh \end{aligned}\right\} \qquad (8).$$

41. Gradient of a scalar function. The three-parametric gradient of a scalar point-function, ϕ, is a vector quantity, whose direction is normal to the surface $\phi=$ const. in the sense of ϕ increasing, and whose magnitude is the distance-rate of increase of ϕ in that direction. Let S and S' be two adjacent level-surfaces corresponding to the values ϕ and $\phi+\delta\phi$ of the function, where $\delta\phi$ is positive. Let PQ be an element of an orthogonal trajectory of the level-surfaces, intercepted between S and S', s the arc-length of this trajectory measured from a fixed point, and δs the length of PQ. Then the gradient of ϕ at P has the direction of the tangent at P to the orthogonal trajectory, and magnitude equal to the limiting value of $\delta\phi/\delta s$ as δs tends to zero. This vector is denoted by $\nabla\phi$ or grad ϕ. From the definition it is clearly independent of any choice of parameters u, v, w, and is a vector point-function.

As in the case of the two-parametric gradient already considered, it follows that the rate of increase of ϕ in any other direction is

the resolved part of $\nabla\phi$ in that direction. Or, if $\mathbf{c}$ is a unit vector, $\mathbf{c}\cdot\nabla\phi$ is the derivative of ϕ in the direction of $\mathbf{c}$. And, as in the earlier case, the change $d\phi$ in the function, due to an infinitesimal displacement $d\mathbf{r}$ in space, is given by*

$$d\phi = d\mathbf{r}\cdot\nabla\phi.$$

Although $\nabla\phi$ is independent of any choice of coordinates, it is frequently convenient to express its value in terms of chosen parameters. To find its expression in terms of the oblique curvilinear coordinates u, v, w, consider two infinitesimal displacements $d\mathbf{r}$ and $\delta\mathbf{r}$ on the surface $\phi(u, v, w) = \text{const.}$ Then if (du, dv, dw) and $(\delta u, \delta v, \delta w)$ are the variations of the parameters corresponding to these displacements, we must have

$$\phi_1 du + \phi_2 dv + \phi_3 dw = 0,$$

and

$$\phi_1 \delta u + \phi_2 \delta v + \phi_3 \delta w = 0.$$

Consequently

$$\frac{\phi_1}{dv\,\delta w - \delta v\,dw} = \frac{\phi_2}{dw\,\delta u - \delta w\,du} = \frac{\phi_3}{du\,\delta v - \delta u\,dv}.$$

Now the normal to the surface $\phi = \text{const.}$ is parallel to the vector $d\mathbf{r}\times\delta\mathbf{r}$. But this vector has the value

$$(dv\,\delta w - \delta v\,dw)\,\mathbf{r}_2\times\mathbf{r}_3 + (dw\,\delta u - \delta w\,du)\,\mathbf{r}_3\times\mathbf{r}_1 + \ldots,$$

and is therefore parallel to the vector $\mathbf{V}$ defined by

$$\mathbf{V} = \phi_1\mathbf{r}_2\times\mathbf{r}_3 + \phi_2\mathbf{r}_3\times\mathbf{r}_1 + \phi_3\mathbf{r}_1\times\mathbf{r}_2.$$

Also the resolved part of this vector in the direction of $\mathbf{r}_1$ is equal to

$$\frac{1}{\sqrt{a}}\mathbf{r}_1\cdot\mathbf{V} \quad \text{or} \quad \frac{p}{\sqrt{a}}\phi_1,$$

which is p times the derivative of ϕ in this direction. Hence the required expression for the gradient is

$$\nabla\phi = \frac{1}{p}(\phi_1\mathbf{r}_2\times\mathbf{r}_3 + \phi_2\mathbf{r}_3\times\mathbf{r}_1 + \phi_3\mathbf{r}_1\times\mathbf{r}_2) \quad\ldots\ldots\ldots(9).$$

Inserting the values of these cross products found above, we have the alternative expression

$$\nabla\phi = \frac{1}{D}\{(A\phi_1 + H\phi_2 + G\phi_3)\,\mathbf{r}_1 + (H\phi_1 + B\phi_2 + F\phi_3)\,\mathbf{r}_2 + (G\phi_1 + F\phi_2 + C\phi_3)\,\mathbf{r}_3\} \ldots(10).$$

* Vol. I, Art. 114.

In particular, the gradients of the functions u, v, w are

$$\left.\begin{aligned}\nabla u &= \frac{1}{p}\mathbf{r}_2\times\mathbf{r}_3 = \frac{1}{D}(A\mathbf{r}_1 + H\mathbf{r}_2 + G\mathbf{r}_3)\\ \nabla v &= \frac{1}{p}\mathbf{r}_3\times\mathbf{r}_1 = \frac{1}{D}(H\mathbf{r}_1 + B\mathbf{r}_2 + F\mathbf{r}_3)\\ \nabla w &= \frac{1}{p}\mathbf{r}_1\times\mathbf{r}_2 = \frac{1}{D}(G\mathbf{r}_1 + F\mathbf{r}_2 + C\mathbf{r}_3)\end{aligned}\right\} \quad \text{.........(11).}$$

The three vectors represented in (11) constitute the reciprocal system of vectors to $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$. Let them be denoted by $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ respectively. Then clearly

$$\left.\begin{aligned}&\mathbf{l}\cdot\mathbf{r}_1 = 1, \quad \mathbf{m}\cdot\mathbf{r}_2 = 1, \quad \mathbf{n}\cdot\mathbf{r}_3 = 1\\ \text{while} \quad &\mathbf{l}\cdot\mathbf{r}_2 = \mathbf{l}\cdot\mathbf{r}_3 = \text{etc.} = 0\end{aligned}\right\} \text{............(12).}$$

It is also easily verified that

$$\left.\begin{aligned}\mathbf{l}^2 = \frac{A}{D}, \quad \mathbf{m}^2 = \frac{B}{D}, \quad \mathbf{n}^2 = \frac{C}{D}\\ \mathbf{m}\cdot\mathbf{n} = \frac{F}{D}, \quad \mathbf{n}\cdot\mathbf{l} = \frac{G}{D}, \quad \mathbf{l}\cdot\mathbf{m} = \frac{H}{D}\end{aligned}\right\} \quad \text{.........(13),}$$

which are relations corresponding to (1). And further, it follows from (12) that

$$\left.\begin{aligned}\mathbf{r}_1 &= a\mathbf{l} + h\mathbf{m} + g\mathbf{n}\\ \mathbf{r}_2 &= h\mathbf{l} + b\mathbf{m} + f\mathbf{n}\\ \mathbf{r}_3 &= g\mathbf{l} + f\mathbf{m} + c\mathbf{n}\end{aligned}\right\} \text{.....................(14),}$$

which correspond with (11). The value found above for the gradient of ϕ may be written simply

$$\nabla\phi = \mathbf{l}\frac{\partial\phi}{\partial u} + \mathbf{m}\frac{\partial\phi}{\partial v} + \mathbf{n}\frac{\partial\phi}{\partial w} \quad \text{...............(15).}$$

42. The operator ∇. The last equation may be expressed

$$\nabla\phi = \left(\mathbf{l}\frac{\partial}{\partial u} + \mathbf{m}\frac{\partial}{\partial v} + \mathbf{n}\frac{\partial}{\partial w}\right)\phi,$$

if we interpret the second member according to the distributive law, and regard ∇ as the vectorial differential operator defined by

$$\nabla = \mathbf{l}\frac{\partial}{\partial u} + \mathbf{m}\frac{\partial}{\partial v} + \mathbf{n}\frac{\partial}{\partial w},$$

and such that the result of its operation on a scalar function is a vector.

We have seen that, if $\mathbf{c}$ is a unit vector, $\mathbf{c}\cdot\nabla\phi$ is the derivative of ϕ in the direction of $\mathbf{c}$. Now $\mathbf{c}$ may be expressed in the form

$$\mathbf{c} = l\mathbf{r}_1 + m\mathbf{r}_2 + n\mathbf{r}_3.$$

Then, in virtue of (12),

$$\mathbf{c}\cdot\nabla\phi = l\frac{\partial\phi}{\partial u} + m\frac{\partial\phi}{\partial v} + n\frac{\partial\phi}{\partial w}$$

$$= \left(l\frac{\partial}{\partial u} + m\frac{\partial}{\partial v} + n\frac{\partial}{\partial w}\right)\phi \quad\ldots\ldots\ldots\ldots(16),$$

the operator in brackets being interpreted according to the distributive law. This operator is the formal scalar product of $\mathbf{c}$ and ∇, and is written $\mathbf{c}\cdot\nabla$. The equation (16) then becomes

$$\mathbf{c}\cdot\nabla\phi = (\mathbf{c}\cdot\nabla)\,\phi.$$

The brackets are therefore unnecessary.

This operator $\mathbf{c}\cdot\nabla$ may also be applied to a vector function $\mathbf{s}$, being interpreted

$$\mathbf{c}\cdot\nabla\mathbf{s} = l\frac{\partial\mathbf{s}}{\partial u} + m\frac{\partial\mathbf{s}}{\partial v} + n\frac{\partial\mathbf{s}}{\partial w} \quad\ldots\ldots\ldots\ldots\ldots(17).$$

If $\mathbf{c}$ is a unit vector this expression gives the derivative of $\mathbf{s}$ in the direction of $\mathbf{c}$. But whether $\mathbf{c}$ is a unit vector or not, $\mathbf{c}\cdot\nabla\mathbf{s}$ is to be interpreted according to (17).

From (15) it follows immediately that the gradients of a sum and of a product are given by the ordinary rules of differentiation. Thus

$$\nabla(\phi + \psi + \ldots) = \nabla\phi + \nabla\psi + \ldots,$$

and

$$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi \quad\ldots\ldots\ldots\ldots\ldots\ldots(18).$$

Also, as in the case of the two-parametric gradient, if $f(\theta, \phi, \ldots)$ is a function of several point-functions,

$$\nabla f = \frac{\partial f}{\partial\theta}\nabla\theta + \frac{\partial f}{\partial\phi}\nabla\phi + \ldots \quad\ldots\ldots\ldots\ldots(18').$$

43. Divergence and rotation of a vector. By operating on a vector function $\mathbf{s}$ with the operator ∇ in different ways, we obtain the *divergence* and the *rotation* of that vector. The former is a scalar function denoted by div $\mathbf{s}$ or $\nabla\cdot\mathbf{s}$, and defined by

$$\operatorname{div}\mathbf{s} = \nabla\cdot\mathbf{s} = \mathbf{l}\cdot\frac{\partial\mathbf{s}}{\partial u} + \mathbf{m}\cdot\frac{\partial\mathbf{s}}{\partial v} + \mathbf{n}\cdot\frac{\partial\mathbf{s}}{\partial w} \quad\ldots\ldots\ldots(19).$$

The other is a vector function, denoted by rot $\mathbf{s}$, curl $\mathbf{s}$ or $\nabla \times \mathbf{s}$, and defined by

$$\operatorname{rot} \mathbf{s} = \nabla \times \mathbf{s} = \mathbf{l} \times \frac{\partial \mathbf{s}}{\partial u} + \mathbf{m} \times \frac{\partial \mathbf{s}}{\partial v} + \mathbf{n} \times \frac{\partial \mathbf{s}}{\partial w} \quad \ldots\ldots(20).$$

That these functions are differential invariants of $\mathbf{s}$, being independent of the choice of parameters, will be shown presently in connection with the transformation of certain integrals; and it will then be seen how div $\mathbf{s}$ and rot $\mathbf{s}$ may be defined without the use of coordinates.

We may find an expression for div $\mathbf{s}$ in terms of the components of $\mathbf{s}$ in the directions of the parametric curves. Suppose that

$$\mathbf{s} = X\mathbf{r}_1 + Y\mathbf{r}_2 + Z\mathbf{r}_3.$$

Then, considering the divergence of the first component, we have

$$\begin{aligned} \operatorname{div}(X\mathbf{r}_1) &= \frac{1}{p}\left\{\mathbf{r}_2 \times \mathbf{r}_3 \cdot \frac{\partial}{\partial u}(X\mathbf{r}_1) + \mathbf{r}_3 \times \mathbf{r}_1 \cdot \frac{\partial}{\partial v}(X\mathbf{r}_1) + \ldots\right\} \\ &= \frac{1}{p}\left\{pX_1 + X\frac{\partial}{\partial u}(\mathbf{r}_2 \times \mathbf{r}_3 \cdot \mathbf{r}_1)\right\} \\ &= \frac{1}{p}\frac{\partial}{\partial u}(pX). \end{aligned}$$

Similarly it may be shown that

$$\operatorname{div}(Y\mathbf{r}_2) = \frac{1}{p}\frac{\partial}{\partial v}(pY),$$

and

$$\operatorname{div}(Z\mathbf{r}_3) = \frac{1}{p}\frac{\partial}{\partial w}(pZ).$$

Combining these results we have the required formula

$$\operatorname{div} \mathbf{s} = \frac{1}{p}\left[\frac{\partial}{\partial u}(pX) + \frac{\partial}{\partial v}(pY) + \frac{\partial}{\partial w}(pZ)\right] \quad \ldots\ldots(21),$$

which will be frequently employed in the following argument.

The function rot $\mathbf{s}$, as defined by (20), may be expressed as the sum of components in the directions of the parametric curves. For, the second member of (20) is equivalent to

$$\Sigma \frac{1}{p}(\mathbf{r}_2 \times \mathbf{r}_3) \times \mathbf{s}_1 = \frac{1}{p}\Sigma\left[(\mathbf{r}_2 \cdot \mathbf{s}_1)\mathbf{r}_3 - (\mathbf{r}_3 \cdot \mathbf{s}_1)\mathbf{r}_2\right].$$

In this sum the coefficient of $\mathbf{r}_1$ is

$$\frac{1}{p}(\mathbf{r}_3 \cdot \mathbf{s}_2 - \mathbf{r}_2 \cdot \mathbf{s}_3) = \frac{1}{p}\left[\frac{\partial}{\partial v}(\mathbf{r}_3 \cdot \mathbf{s}) - \frac{\partial}{\partial w}(\mathbf{r}_2 \cdot \mathbf{s})\right],$$

and similarly for the coefficients of $\mathbf{r}_2$ and $\mathbf{r}_3$. Combining the results we have the formula

$$\operatorname{rot}\mathbf{s} = \frac{1}{p}\left\{\frac{\partial}{\partial v}(\mathbf{r}_3\cdot\mathbf{s}) - \frac{\partial}{\partial w}(\mathbf{r}_2\cdot\mathbf{s})\right\}\mathbf{r}_1 + \frac{1}{p}\left\{\frac{\partial}{\partial w}(\mathbf{r}_1\cdot\mathbf{s}) - \frac{\partial}{\partial u}(\mathbf{r}_3\cdot\mathbf{s})\right\}\mathbf{r}_2 + \ldots \quad \ldots(22).$$

If $\mathbf{s}$ is expressed as the sum of components in the directions of $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ in the form

$$\mathbf{s} = P\mathbf{l} + Q\mathbf{m} + R\mathbf{n},$$

(22) becomes simply

$$\operatorname{rot}\mathbf{s} = \frac{1}{p}\left[(R_2 - Q_3)\,\mathbf{r}_1 + (P_3 - R_1)\,\mathbf{r}_2 + (Q_1 - P_2)\,\mathbf{r}_3\right] \quad \ldots(23).$$

Suppose that the vector point-function $\mathbf{s}$ in space is everywhere tangential to the surface $w = \text{const}$. Then there is a useful relation between the three-parametric divergence of $\mathbf{s}$, and the two-parametric divergence of a certain multiple of $\mathbf{s}$ relative to the surface $w = \text{const}$. Let this divergence be denoted for the moment by Div $\mathbf{s}$, and the former by div $\mathbf{s}$. Then since

$$\mathbf{s} = X\mathbf{r}_1 + Y\mathbf{r}_2,$$

it follows that

$$\operatorname{div}\mathbf{s} = \frac{1}{p}\left[\frac{\partial}{\partial u}(pX) + \frac{\partial}{\partial v}(pY)\right] = \frac{1}{p}\sqrt{C}\,\operatorname{Div}(p\mathbf{s}/\sqrt{C}),$$

by (12) of Art. 2, since, on the surface $w = \text{const.}$, $H^2 = ab - h^2 = C$. We may express this relation

$$\operatorname{div}\mathbf{s} = \frac{1}{\psi}\operatorname{Div}(\psi\mathbf{s}) \quad \ldots\ldots\ldots\ldots(24),$$

where

$$\psi^2 = \frac{p^2}{C} = \frac{1}{(\nabla w)^2}.$$

This function ψ is the "distance function" for the family of surfaces $w = \text{const}$. It will play an important part in the following chapter.

44. Formulae of expansion. The formulae of expansion proved in Art. 3 are true also for the three-parametric gradient, divergence and rotation, and may be established in the same

manner, using the definitions (15), (19) and (20) of these invariants. It will be found convenient to record them here for reference:

$$\operatorname{div}(\phi\mathbf{s}) = \nabla\phi\cdot\mathbf{s} + \phi\operatorname{div}\mathbf{s} \quad\text{.........................(25),}$$

$$\operatorname{rot}(\phi\mathbf{s}) = \nabla\phi\times\mathbf{s} + \phi\operatorname{rot}\mathbf{s} \quad\text{.......................(26),}$$

$$\operatorname{div}(\mathbf{s}\times\mathbf{t}) = \mathbf{t}\cdot\operatorname{rot}\mathbf{s} - \mathbf{s}\cdot\operatorname{rot}\mathbf{t} \quad\text{.......................(27),}$$

$$\operatorname{rot}(\mathbf{s}\times\mathbf{t}) = \mathbf{t}\cdot\nabla\mathbf{s} - \mathbf{s}\cdot\nabla\mathbf{t} + \mathbf{s}\operatorname{div}\mathbf{t} - \mathbf{t}\operatorname{div}\mathbf{s} \quad\text{......(28),}$$

$$\operatorname{grad}(\mathbf{s}\cdot\mathbf{t}) = \mathbf{t}\cdot\nabla\mathbf{s} + \mathbf{s}\cdot\nabla\mathbf{t} + \mathbf{t}\times\operatorname{rot}\mathbf{s} + \mathbf{s}\times\operatorname{rot}\mathbf{t} \quad\text{...(29).}$$

If $\mathbf{t}$ is a vector of constant magnitude, though variable direction, (29) gives

$$\nabla\mathbf{t}^2 = 2\mathbf{t}\cdot\nabla\mathbf{t} + 2\mathbf{t}\times\operatorname{rot}\mathbf{t}.$$

The first member is zero, since $\mathbf{t}^2$ is constant. Consequently, for a *vector of constant magnitude,* we have the identity

$$\mathbf{t}\cdot\nabla\mathbf{t} = -\mathbf{t}\times\operatorname{rot}\mathbf{t} \quad\text{....................(30).}$$

45. Differential invariants of the second order. If ϕ and $\mathbf{s}$ are point-functions, $\nabla\phi$, div $\mathbf{s}$ and rot $\mathbf{s}$ are also in general point-functions. The first and third possess divergence and rotation, and the second a gradient. Consider then the functions

$$\left.\begin{aligned} \operatorname{div}\operatorname{grad}\phi &= \nabla\cdot\nabla\phi \\ \operatorname{rot}\operatorname{grad}\phi &= \nabla\times\nabla\phi \\ \operatorname{div}\operatorname{rot}\mathbf{s} &= \nabla\cdot\nabla\times\mathbf{s} \\ \operatorname{rot}\operatorname{rot}\mathbf{s} &= \nabla\times\nabla\times\mathbf{s} \\ \operatorname{grad}\operatorname{div}\mathbf{s} &= \nabla\nabla\cdot\mathbf{s} \end{aligned}\right\} \quad\text{..............(31),}$$

which are second order differential invariants of ϕ and $\mathbf{s}$. The second and third of these vanish identically. The vanishing of the second follows immediately from (15) and (23), and that of the third from (21) and (22). These two identities

$$\left.\begin{aligned} \operatorname{rot}\operatorname{grad}\phi &= 0 \\ \operatorname{div}\operatorname{rot}\mathbf{s} &= 0 \end{aligned}\right\} \quad\text{....................(32)}$$

are very important. Conversely, it may easily be proved that a vector function whose rotation vanishes identically is the gradient of some scalar function*; while one whose divergence vanishes identically is the rotation of some vector function.

* *Advanced Vector Analysis*, Art. 17.

The first of the functions (31), $\nabla \cdot \nabla\phi$, is the *Laplacian* of ϕ, and is often denoted by $\nabla^2\phi$. The operator ∇^2 is called Laplace's operator, and $\nabla^2\phi = 0$ is Laplace's equation. A point-function which satisfies Laplace's equation within a given region is said to be *harmonic* within that region. If we use the value (10) for $\nabla\phi$, it follows from (21) that

$$\nabla^2\phi = \frac{1}{p}\left[\frac{\partial}{\partial u}\left(\frac{A\phi_1 + H\phi_2 + G\phi_3}{p}\right) + \frac{\partial}{\partial v}\left(\frac{H\phi_1 + B\phi_2 + F\phi_3}{p}\right) + \frac{\partial}{\partial w}\left(\frac{G\phi_1 + F\phi_2 + C\phi_3}{p}\right)\right] \quad \ldots(33),$$

which gives the value of $\nabla^2\phi$ in terms of oblique curvilinear coordinates and the fundamental magnitudes.

The operator ∇^2 may also be applied to a vector function, provided we interpret the result by the formula (33), or its equivalent. Then the fourth of the functions (31) has an alternative expression which is sometimes found useful, viz.

$$\text{rot rot } \mathbf{s} = \text{grad div } \mathbf{s} - \nabla^2\mathbf{s} \quad \ldots\ldots\ldots\ldots\ldots(34).$$

A proof of this transformation will be given in the following Art.

46. Orthogonal coordinates. An important particular case of the preceding theory is that in which the parametric surfaces constitute a triply orthogonal system. Since the parametric curves cut orthogonally we have

$$f = g = h = 0,$$

and consequently

$$F = G = H = 0,$$

$$A = bc, \quad B = ca, \quad C = ab,$$

$$D = p^2 = abc.$$

The formulae (11) become simply

$$\mathbf{l} = \mathbf{r}_1/a, \quad \mathbf{m} = \mathbf{r}_2/b, \quad \mathbf{n} = \mathbf{r}_3/c,$$

so that

$$\nabla\phi = \frac{\mathbf{r}_1}{a}\frac{\partial\phi}{\partial u} + \frac{\mathbf{r}_2}{b}\frac{\partial\phi}{\partial v} + \frac{\mathbf{r}_3}{c}\frac{\partial\phi}{\partial w},$$

and

$$\nabla^2\phi = \frac{1}{\sqrt{abc}}\left[\frac{\partial}{\partial u}\left(\phi_1\sqrt{\frac{bc}{a}}\right) + \frac{\partial}{\partial v}\left(\phi_2\sqrt{\frac{ca}{b}}\right) + \frac{\partial}{\partial w}\left(\phi_3\sqrt{\frac{ab}{c}}\right)\right],$$

while

$$\operatorname{rot}(X\mathbf{r}_1 + Y\mathbf{r}_2 + Z\mathbf{r}_3) = \frac{\mathbf{r}_1}{\sqrt{abc}}\left\{\frac{\partial}{\partial v}(cZ) - \frac{\partial}{\partial w}(bY)\right\}$$
$$+ \frac{\mathbf{r}_2}{\sqrt{abc}}\left\{\frac{\partial}{\partial w}(aX) - \frac{\partial}{\partial u}(cZ)\right\} + \frac{\mathbf{r}_3}{\sqrt{abc}}\left\{\frac{\partial}{\partial u}(bY) - \frac{\partial}{\partial v}(aX)\right\}.$$

We may here also record that, in terms of the magnitudes a, b, c, Lamé's relations of Art. 111, Vol. I, become

$$a_{23} - \frac{a_2 a_3}{2a} = \frac{a_2 b_3}{2b} + \frac{a_3 c_2}{2c} \quad \text{..............(35)},$$

with two similar equations, and

$$\sqrt{ab}\left[\frac{\partial}{\partial v}\left(\frac{a_2}{\sqrt{ab}}\right) + \frac{\partial}{\partial u}\left(\frac{b_1}{\sqrt{ab}}\right)\right] + \frac{a_3 b_3}{2c} = 0 \quad \text{......(36)},$$

with two others to correspond.

Cartesian coordinates, x, y, z, are the commonest of all orthogonal coordinates. These measure actual distance in the directions of the mutually perpendicular axes, the parametric surfaces being three orthogonal systems of parallel planes. The unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in the directions of the coordinate axes are constant, and the position vector, $\mathbf{r}$, of a point is

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

In this case

$$a = b = c = 1, \quad p = 1,$$
$$\mathbf{l} = \mathbf{i}, \quad \mathbf{m} = \mathbf{j}, \quad \mathbf{n} = \mathbf{k}.$$

The gradient and Laplacian of ϕ are

$$\nabla\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z},$$
$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2},$$

while the divergence and rotation of any vector are given by

$$\operatorname{div}(X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}) = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z},$$
$$\operatorname{rot}(X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}) = (Z_2 - Y_3)\,\mathbf{i} + (X_3 - Z_1)\,\mathbf{j} + (Y_1 - X_2)\,\mathbf{k},$$

the suffixes having their usual significance.

By means of these coordinates the relation (34) is easily proved. For

$$\operatorname{rot}\mathbf{s} = \mathbf{i}\times\mathbf{s}_1 + \mathbf{j}\times\mathbf{s}_2 + \mathbf{k}\times\mathbf{s}_3,$$

and therefore, since $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are constant unit vectors,

$$\begin{aligned}\operatorname{rot}\operatorname{rot}\mathbf{s} &= \mathbf{i}\times(\mathbf{i}\times\mathbf{s}_{11}+\mathbf{j}\times\mathbf{s}_{12}+\mathbf{k}\times\mathbf{s}_{13})\\ &\quad+\mathbf{j}\times(\mathbf{i}\times\mathbf{s}_{21}+\mathbf{j}\times\mathbf{s}_{22}+\mathbf{k}\times\mathbf{s}_{23})+\dots\\ &=(\mathbf{i}\cdot\mathbf{s}_{11})\mathbf{i}-\mathbf{s}_{11}+(\mathbf{i}\cdot\mathbf{s}_{12})\mathbf{j}+(\mathbf{i}\cdot\mathbf{s}_{13})\mathbf{k}\\ &\quad+(\mathbf{j}\cdot\mathbf{s}_{12})\mathbf{i}-\mathbf{s}_{22}+(\mathbf{j}\cdot\mathbf{s}_{22})\mathbf{j}+(\mathbf{j}\cdot\mathbf{s}_{23})\mathbf{k}+\dots\\ &=\left(\mathbf{i}\frac{\partial}{\partial u}+\mathbf{j}\frac{\partial}{\partial v}+\mathbf{k}\frac{\partial}{\partial w}\right)(\mathbf{i}\cdot\mathbf{s}_1+\mathbf{j}\cdot\mathbf{s}_2+\mathbf{k}\cdot\mathbf{s}_3)-\nabla^2\mathbf{s}\\ &=\nabla\operatorname{div}\mathbf{s}-\nabla^2\mathbf{s},\end{aligned}$$

as required.

Ex. If $\mathbf{n}$ is a vector of constant magnitude, show that

$$\mathbf{n}\cdot\nabla^2\mathbf{n}+\left(\frac{\partial\mathbf{n}}{\partial x}\right)^2+\left(\frac{\partial\mathbf{n}}{\partial y}\right)^2+\left(\frac{\partial\mathbf{n}}{\partial z}\right)^2=0,$$

and hence deduce the relation

$$\mathbf{n}\cdot\nabla^2\mathbf{n}+(\operatorname{rot}\mathbf{n})^2+(\operatorname{div}\mathbf{n})^2=2\Sigma\mathbf{i}\cdot\left(\frac{\partial\mathbf{n}}{\partial y}\times\frac{\partial\mathbf{n}}{\partial z}\right).$$

47. Semi-orthogonal coordinates. In dealing with an arbitrary family of surfaces, it is not always possible to choose orthogonal curvilinear coordinates with the given family as one family of parametric surfaces; for not every family of surfaces is a Lamé family (Vol. I, Art. 112). We can, however, use what may be called *semi-orthogonal coordinates*, with the given family as one set of parametric surfaces, say $w=$ const. Such a system of surfaces may be obtained as follows. Take any surface, S, of the given family, and any set of parametric curves on it, say $u=$const., $v=$const. Consider now the congruence of orthogonal trajectories of the given family of surfaces. All the trajectories which intersect a curve $u=$ const. on S lie on a surface, which may be designated $u=$ const. Similarly all the trajectories which intersect a curve $v=$ const. on S lie on a surface $v=$ const. Then any curve $u=$ const., $v=$ const. is an orthogonal trajectory of the given family of surfaces $w=$ const., so that

$$\mathbf{r}_2\cdot\mathbf{r}_3=0,\quad \mathbf{r}_3\cdot\mathbf{r}_1=0 \quad\dots\dots\dots\dots(37).$$

That is to say, with the above notation for oblique curvilinear coordinates, we have the simplification

$$f=0,\quad g=0,$$

and therefore
$$F = 0, \quad G = 0, \quad H = -ch,$$
$$A = bc, \quad B = ca, \quad C = ab - h^2,$$
$$p^2 = D = cC = c(ab - h^2).$$

The unit normal to a surface $w = \text{const.}$ is $\mathbf{r}_3/\sqrt{c}$, while the vectors $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$, in virtue of (11), have the values

$$(b\mathbf{r}_1 - h\mathbf{r}_2)/C, \quad (a\mathbf{r}_2 - h\mathbf{r}_1)/C, \quad \mathbf{r}_3/c.$$

For a surface $w = \text{const.}$ the first order magnitudes, relative to the coordinates u, v, are a, b, h. The second order magnitudes may be found as follows. Since $\mathbf{r}_3/\sqrt{c}$ is the unit normal,

$$L = \frac{\mathbf{r}_{11} \cdot \mathbf{r}_3}{\sqrt{c}} = \frac{1}{\sqrt{c}} \left\{ \frac{\partial}{\partial u} (\mathbf{r}_1 \cdot \mathbf{r}_3) - \mathbf{r}_1 \cdot \mathbf{r}_{13} \right\} = -\frac{a_3}{2\sqrt{c}}.$$

Similarly
$$N = -\frac{b_3}{2\sqrt{c}},$$

and
$$M = \frac{\mathbf{r}_{12} \cdot \mathbf{r}_3}{\sqrt{c}} = -\frac{h_3}{2\sqrt{c}},$$

as is evident on differentiating the identities (37) with respect to u and v. Consequently the first curvature of a surface $w = \text{const.}$ has the value

$$J = -\frac{ab_3 + ba_3 - 2hh_3}{2(ab - h^2)\sqrt{c}} = -\frac{1}{2\sqrt{c}} \frac{\partial}{\partial w} \log C \quad \text{........(38)},$$

and the second curvature is given by

$$K = \frac{a_3 b_3 - h_3^2}{4c(ab - h^2)} \quad \text{....................(39)}.$$

Ex. If rot $\mathbf{n}$ is the three-parametric rotation of the unit normal to the family $w = \text{const.}$, and Div denotes the two-parametric divergence relative to a surface $w = \text{const.}$, show that

$$\text{Div rot } \mathbf{n} = 0.$$

***48. A differential invariant.** In addition to the differential invariants already considered, there is another invariant of a vector function which may be mentioned here. We proceed to prove the theorem †:

† Given by the author in a paper "On Families of Curves and Surfaces," *Quarterly Journal*, Vol. 50 (1927), p. 354.

If $\mathbf{s}$ *is a vector point-function in space, the scalar triple product of its derivatives in three non-coplanar directions, divided by the scalar triple product of the unit vectors in those directions, is a differential invariant of* $\mathbf{s}$.

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be unit vectors in the given directions. Then, in terms of Cartesian coordinates and the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$, the derivative of $\mathbf{s}$ in the direction of $\mathbf{a}$ is given by

$$\mathbf{a} \cdot \nabla \mathbf{s} = (\mathbf{a} \cdot \mathbf{i})\, \mathbf{s}_1 + (\mathbf{a} \cdot \mathbf{j})\, \mathbf{s}_2 + (\mathbf{a} \cdot \mathbf{k})\, \mathbf{s}_3,$$

and similarly for the derivatives in the directions of $\mathbf{b}$ and $\mathbf{c}$. Hence

$$\begin{aligned}(\mathbf{a} \cdot \nabla \mathbf{s}) \times (\mathbf{b} \cdot \nabla \mathbf{s}) \\ &= \mathbf{i} \cdot (\mathbf{a} \times \mathbf{b})\, \mathbf{s}_2 \times \mathbf{s}_3 + \mathbf{j} \cdot (\mathbf{a} \times \mathbf{b})\, \mathbf{s}_3 \times \mathbf{s}_1 + \mathbf{k} \cdot (\mathbf{a} \times \mathbf{b})\, \mathbf{s}_1 \times \mathbf{s}_2,\end{aligned}$$

and the scalar triple product of the three derivatives has the value

$$\begin{aligned}&(\mathbf{a} \cdot \nabla \mathbf{s}) \times (\mathbf{b} \cdot \nabla \mathbf{s}) \cdot (\mathbf{c} \cdot \nabla \mathbf{s}) \\ &= [\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3] \{(\mathbf{c} \cdot \mathbf{i})\, \mathbf{i} + (\mathbf{c} \cdot \mathbf{j})\mathbf{j} + (\mathbf{c} \cdot \mathbf{k})\, \mathbf{k}\} \cdot (\mathbf{a} \times \mathbf{b}) \\ &= [\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3]\, \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}).\end{aligned}$$

Consequently the quotient of the scalar triple product of the three derivatives by $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ has the value $[\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3]$, which is independent of the directions of $\mathbf{a}, \mathbf{b}, \mathbf{c}$, being the scalar triple product of the derivatives of $\mathbf{s}$ in the directions of the coordinate axes. Thus the theorem is established.

With the notation of oblique curvilinear coordinates, this invariant is equal to

$$\left[\frac{\mathbf{s}_1}{\sqrt{a}}, \frac{\mathbf{s}_2}{\sqrt{b}}, \frac{\mathbf{s}_3}{\sqrt{c}}\right] \div \left[\frac{\mathbf{r}_1}{\sqrt{a}}, \frac{\mathbf{r}_2}{\sqrt{b}}, \frac{\mathbf{r}_3}{\sqrt{c}}\right] = \frac{1}{p}[\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3].$$

If the vector $\mathbf{s}$ is of constant magnitude the invariant is equal to zero. For, in this case, the derivatives of $\mathbf{s}$ are all perpendicular to $\mathbf{s}$, and are therefore coplanar. Consequently $[\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3] = 0$.

Transformation of Integrals

49. Line and surface integrals. Stokes's theorem. Let C be a curve joining the points A, B, and $\mathbf{t}$ the unit tangent to the curve. The tangential line integral of a vector function $\mathbf{s}$ along the curve from A to B is the definite integral of $\mathbf{s} \cdot \mathbf{t}$, the variable of integration being the arc-length s of the curve, and

the limits of integration the values of s corresponding to A and B. We denote it by

$$\int_A^B \mathbf{s}\cdot\mathbf{t}\,ds \quad \text{or} \quad \int_A^B \mathbf{s}\cdot d\mathbf{r}.$$

If the curve C is closed, the value of the integral taken once round the curve in the positive sense is called the *circulation* of $\mathbf{s}$ round C, and is denoted by

$$\int_0 \mathbf{s}\cdot\mathbf{t}\,ds \quad \text{or} \quad \int_0 \mathbf{s}\cdot d\mathbf{r} \quad\ldots\ldots\ldots\ldots\ldots\ldots(40).$$

In particular if $\mathbf{s}=\nabla\phi$, where ϕ is a single-valued scalar function, we have

$$\int_A^B \mathbf{t}\cdot\nabla\phi\,ds = \int_A^B \nabla\phi\cdot d\mathbf{r} = \int_A^B d\phi = \phi_B - \phi_A,$$

and

$$\int_0 \nabla\phi\cdot d\mathbf{r} = 0,$$

as in the two-parametric case. And, as in that case also, if the integral (40) vanishes for every closed curve in the region, $\mathbf{s}$ is the gradient of some scalar function.

Let S be any surface, which may be either open or closed, dS the area of an element, and $\mathbf{n}$ the unit normal. Then $\mathbf{n}\,dS$ is the vector area of the element, and is often denoted by $d\mathbf{S}$. The normal surface integral, or *flux*, of a vector $\mathbf{s}$ over the surface S is the definite integral of $\mathbf{n}\cdot\mathbf{s}$, and is expressed

$$\iint \mathbf{s}\cdot\mathbf{n}\,dS \quad \text{or} \quad \iint \mathbf{s}\cdot d\mathbf{S} \quad\ldots\ldots\ldots\ldots\ldots(41).$$

Consider any closed curve, C, and an open surface, S, bounded by that curve. Let $\mathbf{n}$ be the unit normal to the surface, in the sense which is positive relative to that in which the boundary C is described. Then the *Circulation Theorem* of Vol. I, Art. 124, remains true if the two-parametric invariant rot $\mathbf{s}$ of the vector function $\mathbf{s}$, relative to the surface S, is replaced by the three-parametric invariant for space. In this form the theorem is due to Stokes. To prove it we have only to show that $\mathbf{n}\cdot\text{rot}\,\mathbf{s}$ has the same value for both invariants; and this may be done by choosing semi-orthogonal curvilinear coordinates, with the given surface S as one of the family $w=\text{const}$. If then $\mathbf{s}=P\mathbf{r}_1+Q\mathbf{r}_2+R\mathbf{r}_3$, it

follows from (22) that, for the space differential invariant,

$$\mathbf{n}\cdot\operatorname{rot}\mathbf{s} = \frac{1}{\sqrt{ab-h^2}}\left[\frac{\partial}{\partial u}(hP+bQ) - \frac{\partial}{\partial v}(aP+hQ)\right].$$

But this is also equal to the value of $\mathbf{n}\cdot\operatorname{rot}\mathbf{s}$ in the case of the two-parametric invariant (Vol. I, Art. 124), since, for a surface $w = \text{const.}$, the first order magnitudes are

$$E = a, \quad F = h, \quad G = b.$$

We thus have Stokes's circulation theorem,

$$\iint \mathbf{n}\cdot\operatorname{rot}\mathbf{s}\,dS = \int_{\circ} \mathbf{s}\cdot d\mathbf{r} \quad\text{..................(42)},$$

which may be stated in the form:

The flux of rot **s** *over any open surface is equal to the circulation of* **s** *round the boundary of that surface.*

Applying this theorem to the vector $\phi\mathbf{c}$, where ϕ is a scalar function and $\mathbf{c}$ a constant vector, we find

$$\iint \mathbf{n}\cdot(\nabla\phi\times\mathbf{c})\,dS = \int_{\circ} \phi\mathbf{c}\cdot d\mathbf{r}.$$

And, since this is true for all values of the constant vector $\mathbf{c}$, we have the theorem

$$\iint \mathbf{n}\times\nabla\phi\,dS = \int_{\circ} \phi\,d\mathbf{r} \quad\text{..................(43)}.$$

50. The Divergence Theorem of Gauss. Consider next a closed surface, S, and the region enclosed by it. Let dS be the area of an element of the surface, $\mathbf{n}$ the unit normal directed *outward* from the region, dV the volume of an element of the enclosed space, and $\mathbf{F}$ a vector point-function which, together with its derivative in any direction, is uniform, finite and continuous. Then the Divergence Theorem, due to Gauss, may be stated:

The flux of the function **F** *across the closed surface is equal to the volume-integral of the divergence of* **F** *taken throughout the enclosed space: or*

$$\iint \mathbf{F}\cdot\mathbf{n}\,dS = \iiint \operatorname{div}\mathbf{F}\,dV \quad\text{..............(44)}.$$

Let us write $\mathbf{F}$ in the form

$$\mathbf{F} = X\mathbf{r}_1 + Y\mathbf{r}_2 + Z\mathbf{r}_3,$$

and, using oblique curvilinear coordinates u, v, w, show that each of the three components of $\mathbf{F}$ satisfies the theorem. In dealing with the component $X\mathbf{r}_1$, consider the portion of the region bounded by the surfaces v, $v+dv$, w, $w+dw$ and the elements of S cut off by them at the points N and M. Since the element of volume dV is equal to $p\,du\,dv\,dw$, we have by (18)

$$\iiint \operatorname{div}(X\mathbf{r}_1)\,dV = \iiint \frac{\partial}{\partial u}(pX)\,du\,dv\,dw$$
$$= \iint \Big[pX\Big]_N^M dv\,dw \quad \text{.............(45).}$$

Again, for the vector area of the element of surface at M we have

$$d\mathbf{S} = (\mathbf{r}_1 du + \mathbf{r}_2 dv) \times (\mathbf{r}_1 \delta u + \mathbf{r}_3 dw),$$

where du, δu are the variations of u corresponding to the sides of the element. Hence the element at M contributes to the value of the surface integral the amount

$$X\mathbf{r}_1 \cdot d\mathbf{S} = X\mathbf{r}_1 \cdot (\mathbf{r}_1 du + \mathbf{r}_2 dv) \times (\mathbf{r}_1 \delta u + \mathbf{r}_3 dw)$$
$$= pX\,dv\,dw.$$

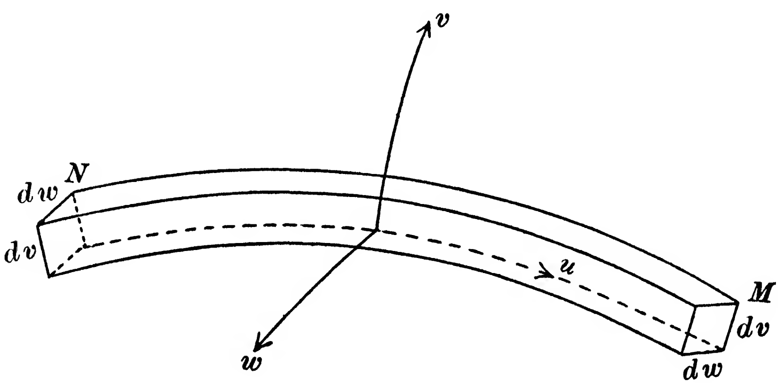

Fig. 5.

Since $d\mathbf{S}$ is everywhere directed outward, the element of surface at N contributes similarly the amount $-pX\,dv\,dw$, the value of pX being taken for that point. Hence, summing for the whole surface,

$$\iint X\mathbf{r}_1 \cdot d\mathbf{S} = \iint \Big[pX\Big]_N^M dv\,dw.$$

Comparing this with (45), we see that the theorem holds for the component $X\mathbf{r}_1$. By a similar proof it may be shown to hold for each of the other components.

From the above theorem we may deduce a definition of div $\mathbf{F}$ independent of any choice of coordinates. For, letting the closed surface S converge to a point P within the enclosed space, we have, for the value of div $\mathbf{F}$ at P,

$$\operatorname{div} \mathbf{F} = \mathrm{Lt} \frac{\iint \mathbf{F} \cdot d\mathbf{S}}{dV} \quad \text{..................(46).}$$

This formula shows that the divergence function, defined by (19), is invariant, and is the limit to which the average value of the outward flux, per unit of enclosed volume, tends as the bounding surface converges to a point.

51. Green's theorem, and others. If in the Divergence Theorem we take the vector function as $\phi\mathbf{c}$, where $\mathbf{c}$ is a constant vector, we have

$$\iiint \mathbf{c} \cdot \nabla\phi \, dV = \iint \phi\mathbf{c} \cdot \mathbf{n} \, dS.$$

And, since this is true for all values of the constant vector $\mathbf{c}$, it follows that

$$\iiint \nabla\phi \, dV = \iint \phi\mathbf{n} \, dS \quad \text{..................(47).}$$

Similarly, by applying the Divergence Theorem to the vector $\mathbf{F} \times \mathbf{c}$, where $\mathbf{c}$ is constant, we obtain the formula

$$\iiint \operatorname{rot} \mathbf{F} \, dV = \iint \mathbf{n} \times \mathbf{F} \, dS \quad \text{..............(48).}$$

This may be used to prove the invariant property of rot $\mathbf{F}$. For, on letting the surface S converge to the point P within it, we have, for the value of rot $\mathbf{F}$ at P,

$$\operatorname{rot} \mathbf{F} = \mathrm{Lt} \frac{\iint \mathbf{n} \times \mathbf{F} \, dS}{dV} \quad \text{..................(49),}$$

giving an alternative definition of rot $\mathbf{F}$. And, since the second member of (49) is independent of the choice of coordinates, it follows that rot $\mathbf{F}$ is invariant.

Again, if ϕ and ψ are scalar point-functions for the region considered, we may apply Gauss's theorem to the vector $\phi\nabla\psi$.

Then, in virtue of (25), we obtain the formula

$$\iiint \nabla\phi\cdot\nabla\psi\, dV = \iint \phi\mathbf{n}\cdot\nabla\psi\, dS - \iiint \phi\nabla^2\psi\, dV \quad \ldots(50).$$

On interchanging ϕ and ψ we have

$$\iiint \nabla\phi\cdot\nabla\psi\, dV = \iint \psi\mathbf{n}\cdot\nabla\phi\, dS - \iiint \psi\nabla^2\phi\, dV \quad \ldots(51).$$

These results are known as *Green's theorem.* Equating the second members of (50) and (51) we have the symmetrical relation

$$\iint (\phi\nabla\psi - \psi\nabla\phi)\cdot\mathbf{n}\, dS = \iiint (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dV \quad \ldots(52),$$

also due to Green.

From (50) it follows that, if ψ is harmonic within the region enclosed by S,

$$\iiint \nabla\phi\cdot\nabla\psi\, dV = \iint \phi\mathbf{n}\cdot\nabla\psi\, dS \quad \ldots\ldots\ldots\ldots(53).$$

And, if ϕ and ψ are both harmonic within this region, (52) becomes

$$\iint (\phi\nabla\psi - \psi\nabla\phi)\cdot\mathbf{n}\, dS = 0 \quad \ldots\ldots\ldots\ldots(54).$$

EXAMPLES V

1. If $\mathbf{r}$ is the position vector of the current point relative to a fixed origin,

$$\operatorname{div}\mathbf{r} = 3, \quad \operatorname{rot}\mathbf{r} = 0.$$

2. Given that $\psi\mathbf{t} = \nabla\phi$, show that

$$\mathbf{t}\cdot\operatorname{rot}\mathbf{t} = 0.$$

3. If r is the distance of the current point from the fixed origin, show that

$$\nabla r^n = nr^{n-2}\mathbf{r},$$

and hence that*
$$\nabla^2 r^n = n(n+1)r^{n-2}.$$

Thus $1/r$ is a solution of Laplace's equation.

More generally, if U is a scalar function of r, show that

$$\nabla^2 U = U'' + \frac{2U'}{r}.$$

4. If r has the same significance, and $\mathbf{a}$, $\mathbf{b}$ are constant, show that

$$\mathbf{a}\cdot\nabla\left(\mathbf{b}\cdot\nabla\frac{1}{r}\right) = \frac{3(\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r})}{r^5} - \frac{(\mathbf{a}\cdot\mathbf{b})}{r^3}.$$

* Cf. the author's *Advanced Vector Analysis*, Arts. 5, 10.

5. Show that Exx. 1, 2 of Art. 4 are true also for the three-parametric invariants.

Similarly Examples I, 3, 4.

6. If $\mathbf{c}$ is a constant unit vector,

$$\mathbf{c}\cdot[\nabla(\mathbf{u}\cdot\mathbf{c})-\nabla\times(\mathbf{u}\times\mathbf{c})]=\operatorname{div}\mathbf{u}.$$

7. Spherical polar coordinates are orthogonal. Show that, with the usual notation (r, θ, ϕ),

$$a=1, \quad b=r^2, \quad c=r^2\sin^2\theta.$$

Similarly, for cylindrical coordinates (r, ϕ, z),

$$a=1, \quad b=r^2, \quad c=1.$$

8. The unit normal to the surface $u=$const. of an oblique system is

$$\frac{p\mathbf{l}}{\sqrt{A}}=\frac{1}{p\sqrt{A}}(A\mathbf{r}_1+H\mathbf{r}_2+G\mathbf{r}_3).$$

Hence show that the first curvature of the surface is given by

$$J=-\frac{1}{p}\left\{\frac{\partial}{\partial u}\left(\frac{A}{\sqrt{A}}\right)+\frac{\partial}{\partial v}\left(\frac{H}{\sqrt{A}}\right)+\frac{\partial}{\partial w}\left(\frac{G}{\sqrt{A}}\right)\right\}.$$

Show also that the second order magnitudes have the values

$$L=\frac{1}{p\sqrt{A}}\{A(h_2-\tfrac{1}{2}b_1)+\tfrac{1}{2}Hb_2+G(f_2-\tfrac{1}{2}b_3)\},$$

$$M=\frac{1}{p\sqrt{A}}\{\tfrac{1}{2}A(g_2+h_3-f_1)+\tfrac{1}{2}Hb_3+\tfrac{1}{2}Gc_2\},$$

$$N=\frac{1}{p\sqrt{A}}\{A(g_3-\tfrac{1}{2}c_1)+H(f_3-\tfrac{1}{2}c_2)+\tfrac{1}{2}Gc_3\}.$$

9. Prove the theorem of Art. 48, using oblique curvilinear coordinates*.

10. With the notation for oblique curvilinear coordinates, show that the second derivatives of $\mathbf{r}$ are given by

$$\left.\begin{aligned}\mathbf{r}_{23}&=\tfrac{1}{2}(g_2+h_3-f_1)\,\mathbf{l}+\tfrac{1}{2}b_3\,\mathbf{m}+\tfrac{1}{2}c_2\,\mathbf{n}\\ \mathbf{r}_{31}&=\tfrac{1}{2}a_3\,\mathbf{l}+\tfrac{1}{2}(f_1-g_2+h_3)\,\mathbf{m}+\tfrac{1}{2}c_1\,\mathbf{n}\\ \mathbf{r}_{12}&=\tfrac{1}{2}a_2\,\mathbf{l}+\tfrac{1}{2}b_1\,\mathbf{m}+\tfrac{1}{2}(f_1+g_2-h_3)\,\mathbf{n}\end{aligned}\right\},$$

and

$$\left.\begin{aligned}\mathbf{r}_{11}&=\tfrac{1}{2}a_1\,\mathbf{l}+(h_1-\tfrac{1}{2}a_2)\,\mathbf{m}+(g_1-\tfrac{1}{2}a_3)\,\mathbf{n}\\ \mathbf{r}_{22}&=(h_2-\tfrac{1}{2}b_1)\,\mathbf{l}+\tfrac{1}{2}b_2\,\mathbf{m}+(f_2-\tfrac{1}{2}b_3)\,\mathbf{n}\\ \mathbf{r}_{33}&=(g_3-\tfrac{1}{2}c_1)\,\mathbf{l}+(f_3-\tfrac{1}{2}c_2)\,\mathbf{m}+\tfrac{1}{2}c_3\,\mathbf{n}\end{aligned}\right\}.$$

Express the first and fourth of these in terms of $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$.

11. Show that the derivatives of $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ are given by formulae of the type

$$\begin{aligned}-D\mathbf{l}_1=&[\tfrac{1}{2}a_1A+(h_1-\tfrac{1}{2}a_2)H+(g_1-\tfrac{1}{2}a_3)G]\,\mathbf{l}\\ &+[\tfrac{1}{2}a_2A+\tfrac{1}{2}b_1H+\tfrac{1}{2}(f_1+g_2-h_3)G]\,\mathbf{m}\\ &+[\tfrac{1}{2}a_3A+\tfrac{1}{2}(f_1-g_2+h_3)H+\tfrac{1}{2}c_1G]\,\mathbf{n}.\end{aligned}$$

* *Quarterly Journal*, loc. cit., p. 354.

12. In the case of orthogonal curvilinear coordinates the second derivatives of $\mathbf{r}$ are given by

$$\mathbf{r}_{11}=\tfrac{1}{2}(a_1\mathbf{l}-a_2\mathbf{m}-a_3\mathbf{n}),$$
$$\mathbf{r}_{23}=\tfrac{1}{2}(b_3\mathbf{m}+c_2\mathbf{n}),$$

and similar formulae, and the derivatives of $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ by

$$\mathbf{l}_1=-\frac{1}{2a}(a_1\mathbf{l}+a_2\mathbf{m}+a_3\mathbf{n}),$$
$$\mathbf{l}_2=-\frac{1}{2a}(a_2\mathbf{l}-b_1\mathbf{m}),$$

and so on.

13. The surfaces $w=$const. of a triply orthogonal system will be isometric provided

$$\frac{\partial^2}{\partial u\,\partial w}\log\frac{ab}{c}=0 \quad\text{and}\quad \frac{\partial^2}{\partial v\,\partial w}\log\frac{ab}{c}=0$$

[$\nabla^2 w/(\nabla w)^2$ must be a function of w only].

14. The unit normal, $\mathbf{n}$, to a surface $w=$const. of a triply orthogonal system is $\mathbf{r}_3/\sqrt{c}$. The first curvature of the surface is

$$J=-\operatorname{div}\mathbf{n}=-\frac{1}{\sqrt{abc}}\frac{\partial}{\partial w}\sqrt{ab}.$$

Also

$$\operatorname{rot}\mathbf{n}=\frac{1}{2p\sqrt{c}}(c_2\mathbf{r}_1-c_1\mathbf{r}_2),$$
$$\mathbf{n}\times\operatorname{rot}\mathbf{n}=-\mathbf{n}\cdot\nabla\mathbf{n}=\frac{1}{2c}\left(\frac{c_1\mathbf{r}_1}{a}+\frac{c_2\mathbf{r}_2}{b}\right)$$
$$=\tfrac{1}{2}\nabla\log c-\frac{c_3\mathbf{r}_3}{2c^2},$$

and

$$\operatorname{rot}(\mathbf{n}\operatorname{div}\mathbf{n}+\mathbf{n}\times\operatorname{rot}\mathbf{n})$$
$$=\frac{1}{2p}\left[\left(\frac{\partial^2}{\partial v\,\partial w}\log\frac{ab}{c}\right)\mathbf{r}_1-\left(\frac{\partial^2}{\partial u\,\partial w}\log\frac{ab}{c}\right)\mathbf{r}_2\right].$$

Hence the condition expressed in Ex. 13 is

$$\operatorname{rot}(\mathbf{n}\operatorname{div}\mathbf{n}+\mathbf{n}\times\operatorname{rot}\mathbf{n})=0.$$

15. Show that the three-parametric divergence of the unit normal, $\mathbf{n}$, to a family of surfaces $w=$const. is equal to the two-parametric divergence of $\mathbf{n}$ for the surface $w=$const.

16. The unit normal, $\mathbf{n}$, for a family of surfaces $\phi=$const. can be expressed in the form $\psi\nabla\phi$, where

$$\psi^2=\frac{1}{(\nabla\phi)^2}.$$

Hence show that the first curvature of a surface of the family is given by

$$J=-(\psi\nabla^2\phi+\mathbf{n}\cdot\nabla\log\psi).$$

Also verify that $\mathbf{n}\cdot\operatorname{rot}\mathbf{n}=0$, and that

$$\mathbf{n}\operatorname{div}\mathbf{n}+\mathbf{n}\times\operatorname{rot}\mathbf{n}=\theta\nabla\phi+\nabla\log\psi,$$

where

$$\theta=\psi^2\nabla^2\phi,$$

and deduce that

$$\operatorname{rot}(\mathbf{n}\operatorname{div}\mathbf{n}+\mathbf{n}\times\operatorname{rot}\mathbf{n})=\nabla\theta\times\nabla\phi.$$

17. Show that, for a family of parallel surfaces, rot $\mathbf{n}=0$ is satisfied identically.

18. If $\mathbf{n}$ is the unit normal to the family $w=$const. of an oblique system, show that

$$\operatorname{rot}\mathbf{n}=\frac{1}{p}\left[\mathbf{r}_1\frac{\partial}{\partial v}\left(\frac{p}{\sqrt{C}}\right)-\mathbf{r}_2\frac{\partial}{\partial u}\left(\frac{p}{\sqrt{C}}\right)\right].$$

19. Considering the unit normal for a family of surfaces as the unit tangent to the orthogonal trajectory, prove that the magnitude of rot $\mathbf{n}$ is the curvature of the orthogonal trajectory, and its direction that of the binormal of this curve.

20. Prove the identity rot $\nabla\phi=0$ by applying* Stokes's theorem to the vector $\nabla\phi$; and the identity div rot $\mathbf{s}=0$ by applying Gauss's theorem to the function rot $\mathbf{s}$.

21. If rot $\mathbf{s}$ vanishes identically, show that $\mathbf{s}$ is the gradient of some scalar function; and, if div $\mathbf{s}$ is identically zero, $\mathbf{s}$ is the rotation of some vector function.

22. Prove the relation

$$\int_0 \phi\nabla\psi\cdot d\mathbf{r}=-\int_0 \psi\nabla\phi\cdot d\mathbf{r}.$$

23. Show that

$$\int_0 \mathbf{r}\cdot d\mathbf{r}=0,$$

and that

$$\int_0 \mathbf{r}\times d\mathbf{r}$$

is twice the vector area of an open surface bounded by the closed curve.

24. Using Gauss's theorem, prove the relations

$$\iiint \mathbf{u}\cdot\nabla\phi\, dV=\iint \phi\mathbf{u}\cdot\mathbf{n}\, dS-\iiint \phi \operatorname{div}\mathbf{u}\, dV,$$

and

$$\iiint \mathbf{v}\cdot\operatorname{rot}\mathbf{u}\, dV=\iint (\mathbf{u}\times\mathbf{v})\cdot\mathbf{n}\, dS+\iiint \mathbf{u}\cdot\operatorname{rot}\mathbf{v}\, dV.$$

25. Prove Gauss's formula

$$4\pi=-\iint \mathbf{n}\cdot\nabla\left(\frac{1}{r}\right)dS,$$

r being the distance measured from a fixed point within the closed surface S.

26. Express Lamé's relation (35) of Art. 46 in the form

$$\frac{\partial^2}{\partial v\,\partial w}\log a=\frac{a_3c_2}{2ac}+\frac{a_2b_3}{2ab}-\frac{a_2a_3}{2a^2}.$$

* *Advanced Vector Analysis*, p. 27.

CHAPTER VI

FAMILIES OF SURFACES

52. First curvature of a surface. In this chapter we shall consider some properties of a singly infinite family of surfaces*. For such a family the unit normal, $\mathbf{n}$, is a point-function in the space occupied by the family; and the same is true of other quantities, such as the first and second curvatures of the surfaces. We have already seen that the first curvature, J, of any surface is the negative of the two-parametric divergence of $\mathbf{n}$ on that surface. We shall first prove that J is also the negative of the three-parametric divergence of $\mathbf{n}$, regarded as a point-function in space.

Take semi-orthogonal coordinates, as in Art. 47, with the given family of surfaces as the parametric surfaces $w = \text{const}$. The reciprocal system of vectors to $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ is

$$(b\mathbf{r}_1 - h\mathbf{r}_2)/C, \quad (a\mathbf{r}_2 - h\mathbf{r}_1)/C, \quad \mathbf{r}_3/c,$$

while the unit normal to the surface $w = \text{const.}$ is $\mathbf{r}_3/\sqrt{c}$. Hence, by (19) of Art. 43, the three-parametric divergence of $\mathbf{n}$ has the value

$$\frac{1}{C}(b\mathbf{r}_1 - h\mathbf{r}_2)\cdot\frac{\partial \mathbf{n}}{\partial u} + \frac{1}{C}(a\mathbf{r}_2 - h\mathbf{r}_1)\cdot\frac{\partial \mathbf{n}}{\partial v} + \frac{\mathbf{n}}{\sqrt{c}}\cdot\frac{\partial \mathbf{n}}{\partial w}.$$

In this expression the last term is zero, since $\mathbf{n}$ is perpendicular to its derivatives. The remainder of the expression represents the two-parametric divergence of $\mathbf{n}$ on the surface $w = \text{const.}$, since for that surface

$$E = a, \quad F = h, \quad G = b, \quad H^2 = C = ab - h^2.$$

* The substance of this chapter is taken from the following papers by the author:

(*a*) "On Families of Curves and Surfaces," *Quarterly Journal*, Vol. 50 (1927), pp. 350–361.

(*b*) "On Families of Surfaces," *Mathematische Annalen*, Bd. 99, S. 473–478.

(*c*) "On Isometric Systems of Curves and Surfaces," *American Journal of Math.*, Vol. 49 (1927), pp. 527–534.

(*d*) "On the Lines of Equidistance of a Family of Surfaces," *Liouville's Journal*, 1929.

(*e*) "On Lamé Families of Surfaces," *Annals of Mathematics*, Vol. 28, pp. 301–308.

Thus the two divergences of $\mathbf{n}$ are equal, and we have the formula

$$J = -\operatorname{div} \mathbf{n} \qquad \text{.........................(1)},$$

which expresses that:

The first curvature of a surface of the family is the negative of the divergence of the unit normal.

Suppose that the equation of the family of surfaces is given in the form

$$\phi(u, v, w) = \text{const.},$$

where u, v, w are any *oblique curvilinear coordinates.* Then the normal to the surface at any point is parallel to $\nabla\phi$; and the unit normal is, by (13) of Art. 41,

$$\frac{p}{\kappa}(\mathbf{l}\phi_1 + \mathbf{m}\phi_2 + \mathbf{n}\phi_3),$$

where

$$\kappa^2 = A\phi_1^2 + B\phi_2^2 + C\phi_3^2 + 2F\phi_2\phi_3 + 2G\phi_3\phi_1 + 2H\phi_1\phi_2.$$

This expression for the unit normal may be put in the form

$$\frac{1}{p\kappa}[(A\phi_1 + H\phi_2 + G\phi_3)\,\mathbf{r}_1 + (H\phi_1 + B\phi_2 + F\phi_3)\,\mathbf{r}_2 + \ldots].$$

Consequently, on taking its divergence, we have*, in virtue of (1),

$$-J = \frac{1}{p}\frac{\partial}{\partial u}\left(\frac{A\phi_1 + H\phi_2 + G\phi_3}{\kappa}\right) + \frac{1}{p}\frac{\partial}{\partial v}\left(\frac{H\phi_1 + B\phi_2 + F\phi_3}{\kappa}\right) + \frac{1}{p}\frac{\partial}{\partial w}\left(\frac{G\phi_1 + F\phi_2 + C\phi_3}{\kappa}\right) \quad \text{...(2)},$$

the sense of $\mathbf{n}$ being taken as that of $\nabla\phi$.

The case of *Cartesian coordinates* is particularly simple. The unit normal for the family $\phi(x, y, z) = \text{const.}$ is

$$\frac{1}{\kappa}\left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right),$$

where

$$\kappa^2 = \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2,$$

and the first curvature is therefore given by

$$-J = \frac{\partial}{\partial x}\left(\frac{1}{\kappa}\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\kappa}\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\kappa}\frac{\partial\phi}{\partial z}\right).$$

A notation independent of coordinates is sometimes convenient.

* *Proc. Royal Soc. Edin.*, Vol. 46, p. 200.

If we are dealing with a family of surfaces $\phi = \text{const.}$, the unit normal at any point may be expressed*

$$\mathbf{n} = \psi \nabla \phi \quad \text{.........................(3)},$$

where

$$\psi^2 = 1/(\nabla\phi)^2 \quad \text{......................(4)}.$$

This function ψ may be called the *distance function* for the family of surfaces. For, since the magnitude of $\nabla\phi$ is the normal derivative of ϕ, it follows from (3) that the distance along the normal between adjacent surfaces ϕ and $\phi + d\phi$ has the value $\psi d\phi$. The lines $\psi = \text{const.}$ on any surface of the family are the *lines of equidistance* for that surface; and the surfaces $\psi = \text{const.}$ are the surfaces of equidistance for the family. Substituting the expression (3) for $\mathbf{n}$ in (1), we obtain, for the first curvature of a surface $\phi = \text{const.}$,

$$J = -(\psi\nabla^2\phi + \nabla\phi \cdot \nabla\psi) \quad \text{.................(5)},$$

or

$$J = -(\psi\nabla^2\phi + \mathbf{n} \cdot \nabla \log \psi) \quad \text{..............(5')}.$$

53. Rotation of n. Parallel surfaces. In the following argument we shall make frequent use of the fact that rot $\mathbf{n}$ is perpendicular to $\mathbf{n}$. This may be seen by taking the rotation of both members of (3). Thus, since rot $\nabla\phi$ vanishes identically, we obtain

$$\text{rot}\, \mathbf{n} = \nabla\psi \times \nabla\phi,$$

showing that rot $\mathbf{n}$ is perpendicular to $\nabla\phi$ and therefore to $\mathbf{n}$. The identity

$$\mathbf{n} \cdot \text{rot}\, \mathbf{n} = 0 \quad \text{.........................(6)}$$

may be interpreted as expressing that the congruence of orthogonal trajectories of the family of surfaces is a normal congruence†.

We may here notice in passing that rot $\mathbf{n}$ vanishes identically for a family of parallel surfaces. For, in the case of parallel surfaces, the unit normal, $\mathbf{n}$, remains unchanged for displacement in its own direction. Consequently $\mathbf{n} \cdot \nabla\mathbf{n} = 0$, and therefore, by (30) of Art. 44, $\mathbf{n} \times \text{rot}\, \mathbf{n} = 0$. And, since rot $\mathbf{n}$ is perpendicular to $\mathbf{n}$, it follows that rot $\mathbf{n} = 0$. Conversely, if rot $\mathbf{n}$ vanishes identically, so does $\mathbf{n} \times \text{rot}\, \mathbf{n}$, and therefore also $\mathbf{n} \cdot \nabla\mathbf{n}$, and the surfaces are parallel. Hence the theorem‡:

* *Math. Annalen*, Bd. 99, S. 473.

† Vol. I, Art. 105. See also Art. 129 of this volume.

‡ *Quarterly Journal*, loc. cit., p. 353.

A necessary and sufficient condition that a singly infinite family of surfaces be parallels is that rot $\mathbf{n}$ *vanish identically.*

For any family of surfaces the condition of parallelism is satisfied at points where rot $\mathbf{n} = 0$. This vector equation is equivalent to two scalar equations, which determine a curve. For, by (22) of Art. 43, since $\mathbf{n}$ is perpendicular to $\mathbf{r}_1$ and $\mathbf{r}_2$, the vanishing of rot $\mathbf{n}$ is equivalent to

$$\frac{\partial}{\partial u}(\mathbf{r}_3 \cdot \mathbf{n}) = 0, \quad \frac{\partial}{\partial v}(\mathbf{r}_3 \cdot \mathbf{n}) = 0,$$

which, in terms of the magnitudes of Art. 40, may be expressed

$$\frac{\partial}{\partial u}\left(\frac{p}{\sqrt{C}}\right) = 0, \quad \frac{\partial}{\partial v}\left(\frac{p}{\sqrt{C}}\right) = 0 \quad \ldots\ldots\ldots\ldots\ldots(7).$$

The curve determined by these equations may be called the *line of parallelism* of the family of surfaces $w = \text{const.}$

54. Second curvature of a surface. We have already seen (Art. 4) that the second curvature of a surface is expressible in terms of the two-parametric differential invariants of $\mathbf{n}$ for that surface by the formula

$$2K = \mathbf{n} \cdot \nabla^2 \mathbf{n} + (\operatorname{div} \mathbf{n})^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(\text{i}).$$

In dealing with a family of surfaces, for which $\mathbf{n}$ is a point-function in the space occupied by the family, we need a formula for K as a three-parametric differential invariant of $\mathbf{n}$. This may be found as follows.

The function $\nabla \cdot \mathbf{n}$ has the same value whether ∇ is two-parametric or three-parametric: but this is not the case with the invariant $\mathbf{n} \cdot \nabla^2 \mathbf{n}$. To find the difference let us employ oblique curvilinear coordinates u, v, w with the given family as parametric surfaces $w = \text{const.}$ Then, for the two-parametric invariant $\mathbf{n} \cdot \nabla^2 \mathbf{n}$ relative to the surface $w = \text{const.}$, we have

$$\begin{aligned}
\mathbf{n} \cdot \nabla^2 \mathbf{n} &= \frac{1}{\sqrt{C}} \mathbf{n} \cdot \left[\frac{\partial}{\partial u}\left(\frac{b\mathbf{n}_1 - h\mathbf{n}_2}{\sqrt{C}}\right) + \frac{\partial}{\partial v}\left(\frac{a\mathbf{n}_2 - h\mathbf{n}_1}{\sqrt{C}}\right)\right] \\
&= \frac{1}{C} \mathbf{n} \cdot (b\mathbf{n}_{11} - 2h\mathbf{n}_{12} + a\mathbf{n}_{22}) \\
&= -\frac{1}{C}(b\mathbf{n}_1^2 - 2h\mathbf{n}_1 \cdot \mathbf{n}_2 + a\mathbf{n}_2^2),
\end{aligned}$$

since $\mathbf{n}$ is perpendicular to its derivatives. If, however, ∇ is the three-parametric operator for space, we have, by (33) of Art. 45,

$$\mathbf{n}\cdot\nabla^2\mathbf{n} = \frac{1}{p}\mathbf{n}\cdot\left[\frac{\partial}{\partial u}\left(\frac{A\mathbf{n}_1 + H\mathbf{n}_2 + G\mathbf{n}_3}{p}\right)\right.$$
$$\left. + \frac{\partial}{\partial v}\left(\frac{H\mathbf{n}_1 + B\mathbf{n}_2 + F\mathbf{n}_3}{p}\right) + \dots\right]$$
$$= \frac{1}{D}\mathbf{n}\cdot(A\mathbf{n}_{11} + B\mathbf{n}_{22} + C\mathbf{n}_{33} + 2F\mathbf{n}_{23} + 2G\mathbf{n}_{31} + 2H\mathbf{n}_{12})$$
$$= -\frac{1}{D}(A\mathbf{n}_1{}^2 + B\mathbf{n}_2{}^2 + C\mathbf{n}_3{}^2 + 2F\mathbf{n}_2\cdot\mathbf{n}_3 + 2G\mathbf{n}_3\cdot\mathbf{n}_1 + 2H\mathbf{n}_1\cdot\mathbf{n}_2).$$

Taking the difference of these two quantities we find that the two-parametric invariant exceeds the three-parametric by

$$\frac{1}{DC}(G^2\mathbf{n}_1{}^2 + F^2\mathbf{n}_2{}^2 + C^2\mathbf{n}_3{}^2 + 2GF\mathbf{n}_1\cdot\mathbf{n}_2 + 2FC\mathbf{n}_2\cdot\mathbf{n}_3 + 2CG\mathbf{n}_3\cdot\mathbf{n}_1)$$
$$= \left[\frac{1}{p\sqrt{C}}(G\mathbf{n}_1 + F\mathbf{n}_2 + C\mathbf{n}_3)\right]^2 = (\mathbf{n}\cdot\nabla\mathbf{n})^2 = (\mathbf{n}\times\operatorname{rot}\mathbf{n})^2.$$

And further, since $\operatorname{rot}\mathbf{n}$ is perpendicular to $\mathbf{n}$, the magnitude of $\mathbf{n}\times\operatorname{rot}\mathbf{n}$ is equal to that of $\operatorname{rot}\mathbf{n}$, so that

$$(\mathbf{n}\times\operatorname{rot}\mathbf{n})^2 = (\operatorname{rot}\mathbf{n})^2.$$

Consequently, if in (i) we replace the two-parametric invariants by three-parametric, we obtain

$$2K = \mathbf{n}\cdot\nabla^2\mathbf{n} + (\operatorname{div}\mathbf{n})^2 + (\operatorname{rot}\mathbf{n})^2 \quad\text{.............(8),}$$

which is the required formula*.

From this we may deduce another, which is sometimes more convenient. Thus (8) is equivalent to

$$2K = \mathbf{n}\cdot(\nabla\operatorname{div}\mathbf{n} - \operatorname{rot}\operatorname{rot}\mathbf{n}) + (\operatorname{div}\mathbf{n})^2 + (\operatorname{rot}\mathbf{n})^2$$
$$= \operatorname{div}(\mathbf{n}\operatorname{div}\mathbf{n} + \mathbf{n}\times\operatorname{rot}\mathbf{n}) \quad\text{.........................(9),}$$

which may be expressed in the alternative form

$$2K = -\operatorname{div}(J\mathbf{n} + \mathbf{n}\cdot\nabla\mathbf{n}) \quad\text{...............(10).}$$

Thus:

The second curvature of a surface of the family is given as a differential invariant of the unit normal by the formula (8), (9) *or* (10).

* *Quarterly Journal*, loc. cit., p. 352.

Since J and K are differential invariants of $\mathbf{n}$ of the first and second orders respectively, they are appropriately called the first and second curvatures.

By inserting in (9) the value of $\mathbf{n}$ given by (3), we obtain an expression for the second curvature of a surface $\phi = \text{const.}$ in terms of ϕ and the distance function ψ. Thus

$$\operatorname{rot} \mathbf{n} = \nabla\psi \times \nabla\phi = -\mathbf{n} \times \nabla \log \psi \qquad \text{(11)},$$

and therefore

$$\mathbf{n} \times \operatorname{rot} \mathbf{n} = \nabla \log \psi - (\mathbf{n} \cdot \nabla \log \psi)\, \mathbf{n} \qquad \text{(12)}.$$

Consequently, in virtue of (5′), we find

$$\begin{aligned} \mathbf{n} \operatorname{div} \mathbf{n} + \mathbf{n} \times \operatorname{rot} \mathbf{n} &= (\psi^2 \nabla^2 \phi)\, \nabla\phi + \nabla \log \psi \\ &= \theta \nabla\phi + \nabla \log \psi \qquad \text{(13)}, \end{aligned}$$

where we have written

$$\theta = \psi^2 \nabla^2 \phi = \frac{\nabla^2 \phi}{(\nabla\phi)^2} \qquad \text{(14)}.$$

From (9) it then follows that†:

The second curvature of a surface of the family $\phi = \text{const.}$ is given by

$$2K = \operatorname{div} (\theta \nabla\phi + \nabla \log \psi) \qquad \text{(15)}.$$

***55. Second curvature** (*continued*). An alternative formula, giving K as a differential invariant of $\mathbf{n}$, may be deduced from (8). Let us employ Cartesian coordinates, with axes in the directions of $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Then, since $\mathbf{n}$ is perpendicular to its derivatives, we have

$$\mathbf{n} \cdot \frac{\partial \mathbf{n}}{\partial x} = 0, \quad \mathbf{n} \cdot \frac{\partial \mathbf{n}}{\partial y} = 0, \quad \mathbf{n} \cdot \frac{\partial \mathbf{n}}{\partial z} = 0,$$

and differentiation of these identities with respect to x, y, z respectively gives

$$\mathbf{n} \cdot \frac{\partial^2 \mathbf{n}}{\partial x^2} = -\left(\frac{\partial \mathbf{n}}{\partial x}\right)^2, \quad \mathbf{n} \cdot \frac{\partial^2 \mathbf{n}}{\partial y^2} = -\left(\frac{\partial \mathbf{n}}{\partial y}\right)^2, \quad \text{etc.}$$

Consequently

$$\mathbf{n} \cdot \nabla^2 \mathbf{n} = -\left(\frac{\partial \mathbf{n}}{\partial x}\right)^2 - \left(\frac{\partial \mathbf{n}}{\partial y}\right)^2 - \left(\frac{\partial \mathbf{n}}{\partial z}\right)^2 \qquad \text{(i)}.$$

Further
$$\operatorname{rot} \mathbf{n} = \mathbf{i} \times \frac{\partial \mathbf{n}}{\partial x} + \mathbf{j} \times \frac{\partial \mathbf{n}}{\partial y} + \mathbf{k} \times \frac{\partial \mathbf{n}}{\partial z},$$

† *Math. Annalen*, loc. cit., S. 474.

and therefore, on squaring both members,

$$(\text{rot}\,\mathbf{n})^2 = \left(\frac{\partial \mathbf{n}}{\partial x}\right)^2 + \left(\frac{\partial \mathbf{n}}{\partial y}\right)^2 + \left(\frac{\partial \mathbf{n}}{\partial z}\right)^2 - (\text{div}\,\mathbf{n})^2$$
$$+ 2\Sigma\left[\left(\mathbf{j}\cdot\frac{\partial \mathbf{n}}{\partial y}\right)\left(\mathbf{k}\cdot\frac{\partial \mathbf{n}}{\partial z}\right) - \left(\mathbf{j}\cdot\frac{\partial \mathbf{n}}{\partial z}\right)\left(\mathbf{k}\cdot\frac{\partial \mathbf{n}}{\partial y}\right)\right].$$

Substituting from this formula and (i) in (8) we find

$$2K = 2\Sigma\,(\mathbf{j}\times\mathbf{k})\cdot\left(\frac{\partial \mathbf{n}}{\partial y}\times\frac{\partial \mathbf{n}}{\partial z}\right).$$

Thus, for any three mutually perpendicular directions, we have the formula*

$$K = \Sigma\mathbf{i}\cdot\left(\frac{\partial \mathbf{n}}{\partial y}\times\frac{\partial \mathbf{n}}{\partial z}\right)\qquad\text{.....................(16)},$$

which may be expressed as follows:

For any choice of three mutually perpendicular directions, the second curvature of a surface of the family is the sum of the resolved parts, in each direction, of the vector product of the derivatives of **n** *in the other two, cyclic order being preserved.*

56. Orthogonal trajectories of the surfaces†. It follows from (11) that the direction of rot **n** is perpendicular to **n** and also to $\nabla\psi$. It is thus tangential to the surface $\phi = \text{const.}$, and also to the surface $\psi = \text{const.}$, and is therefore the direction of the line of equidistance through the point considered. Hence the result:

For a singly infinite family of surfaces, the direction of the line of equidistance through any point is that of the vector rot **n**.

A family of parallel surfaces is characterised by the property that rot **n** vanishes identically. All curves on these surfaces are lines of equidistance.

Consider now the congruence of curves which are the *orthogonal trajectories* of the family of surfaces $\phi = \text{const.}$ The unit tangent, **t**, to the curve at any point is the unit normal, **n**, to the surface through the point. Let κ be the curvature of the curve, s its arc-length, **p** and **b** the unit principal normal and binormal respectively.

* *Ibid.*, S. 477.

† *Ibid.*, S. 475.

Since $\mathbf{n}$ is a vector of constant length,

$$-\mathbf{n}\times \operatorname{rot}\mathbf{n} = \mathbf{n}\cdot\nabla\mathbf{n} = \frac{d\mathbf{t}}{ds} = \kappa\mathbf{p} \quad \text{............(17)},$$

and therefore

$$\operatorname{rot}\mathbf{n} = \mathbf{n}\times(\kappa\mathbf{p}) = \kappa\mathbf{t}\times\mathbf{p} = \kappa\mathbf{b} \quad \text{..............(18)},$$

and we have the theorem:

The magnitude of rot $\mathbf{n}$ *is the curvature of the orthogonal trajectory, and its direction is that of the binormal of this curve.*

From (12) it follows that $\mathbf{n}\times\operatorname{rot}\mathbf{n}$ is the two-parametric gradient of $\log\psi$ on the surface $\phi = \text{const.}$, for it is the component of the space-gradient tangential to that surface. Let us denote this surface-gradient by Grad $\log\psi$ to distinguish it from the other. Then, in virtue of (17), we have the formula

$$-\kappa\mathbf{p} = \operatorname{Grad}\log\psi \quad \text{...................(19)},$$

which may be expressed:

The two-parametric gradient of $\log\psi$ *on any surface of the family is the negative of the vector curvature of the orthogonal trajectory through that point.*

***57. Lines of equidistance on any surface†.** The curves of equidistance on any surface will be parallel curves provided the magnitude of Grad ψ is constant along each curve, that is to say, by (19), provided κ is constant along each of them. Thus:

A necessary and sufficient condition that the lines of equidistance on each surface may be parallels, and the lines of slope of the distance function therefore geodesics, is that the curvature of the orthogonal trajectories of the family of surfaces be constant along each line of equidistance.

In order to examine some further properties of the family of lines of equidistance, $\psi = \text{const.}$, on any one surface, $\phi = \text{const.}$, we observe that the unit tangent $\mathbf{b}$ to these curves may be expressed

$$\mathbf{b} = \rho\operatorname{rot}\mathbf{n},$$

where ρ is the reciprocal of κ, and therefore the radius of curvature of the orthogonal trajectory. The *line of striction* of the family of lines of equidistance is given by the vanishing of the two-parametric divergence of $\mathbf{b}$. Let us, for the moment, distinguish invariants

† *Liouville's Journal*, loc. cit.

relative to the surface $\phi = \text{const.}$, from those relative to space, by using an initial capital for the former. Then the equation of the line of striction is

$$\text{Div}\, \mathbf{b} = 0.$$

Now div rot $\mathbf{n}$ vanishes identically, and therefore, by (24) of Art. 43,

$$0 = \frac{1}{\psi} \text{Div}\, (\psi \text{ rot}\, \mathbf{n}) = \frac{1}{\psi} \text{Div}\, (\psi \kappa \mathbf{b}),$$

which gives, on expansion,

$$\mathbf{b} \cdot (\psi \,\text{Grad}\, \kappa + \kappa \,\text{Grad}\, \psi) + \psi \kappa \,\text{Div}\, \mathbf{b} = 0.$$

The second term is zero, because $\mathbf{b}$ is parallel to the curve $\psi = \text{const.}$ Then, since Div $\mathbf{b}$ vanishes on the line of striction, the equation of this line may be expressed in the form

$$\mathbf{b} \cdot \text{Grad}\, \kappa = 0 \quad \text{.......................(20)},$$

or, since $\mathbf{b}$ is tangential to the surface $\phi = \text{const.}$,

$$\mathbf{b} \cdot \nabla \kappa = 0 \quad \text{..........................(21)}.$$

Thus:

The line of striction is the locus of points at which κ is stationary for displacement along a line of equidistance.

The lines of equidistance will be a family of parallels provided Div $\mathbf{b}$ vanishes identically, that is to say, provided $\mathbf{b} \cdot \nabla \kappa$ is everywhere zero. Hence we are led again to the above theorem on parallel lines of equidistance.

The *moment* of the lines of equidistance, or the torsion τ of the geodesic tangent, is given by

$$\begin{aligned} \tau = \mathbf{b} \cdot \text{Rot}\, \mathbf{b} &= (\rho \text{ rot}\, \mathbf{n}) \cdot \text{Rot}\, (\rho \text{ rot}\, \mathbf{n}) \\ &= \rho^2 (\text{rot}\, \mathbf{n}) \cdot \text{Rot rot}\, \mathbf{n} \quad \text{.......................(22)}, \end{aligned}$$

and *the curves of equidistance will be lines of curvature provided*

$$(\text{rot}\, \mathbf{n}) \cdot \text{Rot rot}\, \mathbf{n} = 0.$$

The geodesic curvature, γ, of a line of equidistance has the value

$$\gamma = \mathbf{n} \cdot \text{Rot}\, \mathbf{b} = \mathbf{n} \cdot \text{rot}\, \mathbf{b},$$

since the normal resolute is the same for these two invariants. Thus

$$\gamma = \mathbf{n} \cdot \text{rot}\, (\rho \text{ rot}\, \mathbf{n}) \quad \text{....................(23)},$$

and the lines of equidistance will be geodesics provided

$$\mathbf{n} \cdot \text{rot}\, (\rho \text{ rot}\, \mathbf{n}) = 0.$$

58. Isometric system of surfaces. The condition that a family of surfaces, $\phi = \text{const.}$, may constitute an isometric system is usually stated in the form that $\nabla^2\phi/(\nabla\phi)^2$ must be a function of ϕ only. It may, however, be very neatly expressed by a differential equation of the second order to be satisfied by the unit normal; and, for this purpose, there is no need to assume that the family of surfaces is a Lamé family, that is to say, forms part of a triply orthogonal system.

Taking the rotation of both members of (13), and remembering that the rotation of the gradient vanishes identically, we have

$$\text{rot}\,(\mathbf{n}\,\text{div}\,\mathbf{n} + \mathbf{n} \times \text{rot}\,\mathbf{n}) = \nabla\theta \times \nabla\phi.$$

This vector will vanish if $\nabla\theta$ is parallel to $\nabla\phi$, which will be the case provided θ is a function of ϕ only. But this is the condition that the family of surfaces should be isometric. Hence the theorem*:

A singly infinite family of surfaces with unit normal $\mathbf{n}$ *will constitute an isometric system provided*

$$\text{rot}\,(\mathbf{n}\,\text{div}\,\mathbf{n} + \mathbf{n} \times \text{rot}\,\mathbf{n}) = 0 \quad\quad (24).$$

This condition may also be expressed

$$\text{rot}\,(\mathbf{n}\,\text{div}\,\mathbf{n} - \mathbf{n}\cdot\nabla\mathbf{n}) = 0.$$

When this relation is satisfied, the vector $\mathbf{n}\,\text{div}\,\mathbf{n} + \mathbf{n} \times \text{rot}\,\mathbf{n}$ is the gradient of a scalar function. Since θ is now a function of ϕ, we may write

$$F = \int \theta\, d\phi.$$

Then (13) becomes

$$\mathbf{n}\,\text{div}\,\mathbf{n} + \mathbf{n} \times \text{rot}\,\mathbf{n} = \nabla\,(F + \log\psi),$$

so that, in virtue of (9), the second curvature of a surface $\phi = \text{const.}$ has the value

$$K = \tfrac{1}{2}\nabla^2\,(F + \log\psi) \quad\quad (25).$$

Thus:

The second curvature for an isometric family of surfaces is the Laplacian of the function $\frac{1}{2}(F + \log\psi)$.

Consider now a triply orthogonal system of surfaces

$$u = \text{const.},\quad v = \text{const.},\quad w = \text{const.}$$

* *Math. Annalen*, loc. cit., S. 476, and *American Journal*, loc. cit., p 532.

The conditions that any one family should be isometric may be neatly expressed in terms of the magnitudes a, b, c (Art. 46). For

$$\nabla^2 w = \frac{1}{\sqrt{abc}} \frac{\partial}{\partial w} \sqrt{\frac{ab}{c}},$$

and
$$(\nabla w)^2 = \frac{1}{c},$$

so that
$$\frac{\nabla^2 w}{(\nabla w)^2} = \sqrt{\frac{c}{ab}} \frac{\partial}{\partial w} \sqrt{\frac{ab}{c}} = \frac{\partial}{\partial w} \log \sqrt{\frac{ab}{c}}.$$

Thus, necessary and sufficient conditions that the family of surfaces $w = \text{const.}$ may be isometric are

$$\frac{\partial^2}{\partial u \partial w} \log \frac{ab}{c} = 0, \quad \frac{\partial^2}{\partial v \partial w} \log \frac{ab}{c} = 0.$$

Similarly the surfaces $v = \text{const.}$ will be isometric provided

$$\frac{\partial^2}{\partial u \partial v} \log \frac{ac}{b} = 0, \quad \frac{\partial^2}{\partial w \partial v} \log \frac{ac}{b} = 0.$$

Consequently, if both families are isometric, it follows from the second and fourth of these equations that

$$\frac{\partial^2}{\partial v \partial w} \log a = 0, \quad \frac{\partial^2}{\partial v \partial w} \log \frac{b}{c} = 0.$$

Now, on a surface $u = \text{const.}$, v and w are current parameters; and the last equation shows that the parametric curves on this surface, which are also lines of curvature by Dupin's theorem*, form an isometric system. Hence†:

If two families of a triply orthogonal system of surfaces are isometric, the lines of curvature on any surface of the other family constitute an isometric system of curves.

59. Family of Weingarten surfaces‡. Consider a singly infinite family of surfaces, each of which is a W-surface. The unit normal, **n**, and the values of J and K are point-functions in the space occupied by the surfaces. The gradients of J and K on any surface are then the components of ∇J and ∇K tangential to that surface. Hence, in order that the lines $J = \text{const.}$ on any surface

* Vol. I, Art. 109.

† *American Journal*, loc. cit., p. 533.

‡ *Math. Zeitschrift*, Bd. 29, S. 298.

should coincide with the lines $K = \text{const.}$, it is necessary and sufficient that $\mathbf{n}$, ∇J and ∇K be coplanar. Thus:

A necessary and sufficient condition that a singly infinite family of surfaces may consist of W-surfaces is expressed by the equation

$$\mathbf{n} \cdot (\nabla J \times \nabla K) = 0 \qquad \text{(26)}.$$

Substituting the values of J and K given by (1) and (8), we may express this condition

$$\mathbf{n} \cdot (\nabla \operatorname{div} \mathbf{n}) \times \nabla [\mathbf{n} \cdot \nabla^2 \mathbf{n} + (\operatorname{rot} \mathbf{n})^2] = 0 \qquad \text{(27)}.$$

Or, if the values of the curvatures given by (5) and (15) are substituted in (26), we obtain a necessary and sufficient condition that the surfaces $\phi = \text{const.}$ may be a family of W-surfaces.

In the family of surfaces just considered, the functional relation between J and K might vary from one surface to another. All that we have supposed is that the lines $J = \text{const.}$ coincide with the lines $K = \text{const.}$ on each surface. If, however, for the whole space occupied by the surfaces there is a definite functional relation between J and K, then the *surfaces* $J = \text{const.}$ coincide with the surfaces $K = \text{const.}$, and consequently ∇J is parallel to ∇K. The condition necessary for this property may be expressed

$$\nabla J \times \nabla K = 0 \qquad \text{(28)}.$$

In order to include the cases in which J or K is constant, we may state the theorem:

A necessary and sufficient condition that, for a family of surfaces, there should be a definite functional relation between the principal curvatures at a point, is expressed by the equation (28).

This condition is equivalent to two scalar equations, while (26) is only one. The values of J and K given above may be substituted in (28), and this condition expressed in terms of either $\mathbf{n}$ or ϕ.

Lamé Families of Surfaces*

60. Equation of condition. We have seen that an arbitrary family of surfaces, $\phi = \text{const.}$, does not in general form part of a triply orthogonal system†. In order that it may do so, the function ϕ must satisfy a certain differential equation of the third order.

* *Annals of Mathematics*, loc. cit. † Vol. I, Art. 112.

We shall now examine the question from a different point of view, proving that the unit normal for a Lamé family of surfaces satisfies the differential equation

$$\operatorname{div}(\mathbf{n}\cdot\nabla \operatorname{rot}\mathbf{n})=0.$$

This equation is one of the second order in $\mathbf{n}$, as the third derivatives vanish identically. Other related forms for this condition will be deduced, and certain consequences of these equations will be found important.

Let us choose orthogonal curvilinear coordinates, u, v, w, with the given family of surfaces as parametric surfaces $w=\text{const.}$ Then, with the notation of Art. 46, the unit normal is $\mathbf{r}_3/\sqrt{c}$, and its rotation is given by

$$\operatorname{rot}\mathbf{n}=\frac{1}{2p\sqrt{c}}(c_2\mathbf{r}_1-c_1\mathbf{r}_2),$$

where $p=\sqrt{abc}$. The derivative of this rotation in the direction of $\mathbf{n}$ is therefore

$$\mathbf{n}\cdot\nabla\operatorname{rot}\mathbf{n}=\frac{1}{2\sqrt{c}}\frac{\partial}{\partial w}\left[\frac{1}{p\sqrt{c}}(c_2\mathbf{r}_1-c_1\mathbf{r}_2)\right].$$

Now, with the present notation, the values of the second derivatives $\mathbf{r}_{13}$ and $\mathbf{r}_{23}$, found in Vol. I, Art. 110, are

$$\mathbf{r}_{13}=\frac{1}{2}\left(\frac{a_3}{a}\mathbf{r}_1+\frac{c_1}{c}\mathbf{r}_3\right),\quad \mathbf{r}_{23}=\frac{1}{2}\left(\frac{b_3}{b}\mathbf{r}_2+\frac{c_2}{c}\mathbf{r}_3\right).$$

Using these relations we may express the above equation

$$\mathbf{n}\cdot\nabla\operatorname{rot}\mathbf{n}=\frac{1}{2p}\left(\frac{\partial^2}{\partial v\partial w}\log c-\frac{c_2b_3}{2bc}\right)\mathbf{r}_1$$
$$-\frac{1}{2p}\left(\frac{\partial^2}{\partial u\partial v}\log c-\frac{c_1a_3}{2ac}\right)\mathbf{r}_2.$$

Now in the divergence of this function, as given by (21) of Art. 43, the derivatives of the third order vanish identically, so that

$$\operatorname{div}(\mathbf{n}\cdot\nabla\operatorname{rot}\mathbf{n})=\frac{1}{4p}\left[\frac{\partial}{\partial v}\left(\frac{c_1a_3}{ac}\right)-\frac{\partial}{\partial u}\left(\frac{c_2b_3}{bc}\right)\right].$$

Then, using Lamé's relations (35) of Art. 46, which may be written in the form

$$\frac{\partial^2}{\partial v\partial w}\log a=\frac{a_3c_2}{2ac}+\frac{a_2b_3}{2ab}-\frac{a_2a_3}{2a^2},$$

we find that the above expression vanishes identically. Hence the theorem:

The unit normal for a Lamé family of surfaces satisfies the differential equation

$$\operatorname{div}(\mathbf{n}\cdot\nabla\operatorname{rot}\mathbf{n})=0 \quad \text{.................(29).}$$

It follows immediately that *a family of parallel surfaces is a Lamé family,* since for such a family $\operatorname{rot}\mathbf{n}$ vanishes identically.

The equation (29) is of the second order in the derivatives of $\mathbf{n}$, for the third derivatives disappear. This may be shown by transforming the equation as follows. Using the formulae of Art. 44, we easily verify the identity

$$\operatorname{rot}(\mathbf{n}\operatorname{div}\mathbf{n}+\mathbf{n}\times\operatorname{rot}\mathbf{n})=(\nabla\operatorname{div}\mathbf{n})\times\mathbf{n}+(\operatorname{rot}\mathbf{n})\cdot\nabla\mathbf{n}-\mathbf{n}\cdot\nabla\operatorname{rot}\mathbf{n}.$$

The divergence of the first member vanishes identically, and therefore

$$\begin{aligned}\operatorname{div}(\mathbf{n}\cdot\nabla\operatorname{rot}\mathbf{n})&=\operatorname{div}[(\operatorname{rot}\mathbf{n})\cdot\nabla\mathbf{n}-\mathbf{n}\times\nabla\operatorname{div}\mathbf{n}]\\&=\operatorname{div}[(\operatorname{rot}\mathbf{n})\cdot\nabla\mathbf{n}]-(\operatorname{rot}\mathbf{n})\cdot\nabla\operatorname{div}\mathbf{n}.\end{aligned}$$

Since this last expression involves no derivatives of $\mathbf{n}$ higher than the second, it follows that:

The unit normal for a Lamé family of surfaces satisfies a differential equation of the second order, which may be expressed in the alternative forms

$$\operatorname{div}[(\operatorname{rot}\mathbf{n})\cdot\nabla\mathbf{n}-\mathbf{n}\times\nabla\operatorname{div}\mathbf{n}]=0 \quad \text{.........(30),}$$

or

$$\operatorname{div}[(\operatorname{rot}\mathbf{n})\cdot\nabla\mathbf{n}]=(\operatorname{rot}\mathbf{n})\cdot\nabla\operatorname{div}\mathbf{n} \quad \text{.........(31).}$$

If the equation of the family of surfaces is given in the form $\phi=\text{const.}$, the unit normal is $\psi\nabla\phi$. Substituting this value of $\mathbf{n}$ in (29), (30) or (31), we obtain the condition that $\phi=\text{const.}$ may be a Lamé family, expressed as a differential equation to be satisfied by ϕ.

61. Orthogonal trajectories of the surfaces. Consider the congruence of curves which cut the Lamé family of surfaces orthogonally. The unit tangent, $\mathbf{t}$, to the curve at any point is the unit normal, $\mathbf{n}$, to the surface at that point. Let κ, τ be the curvature and torsion of the curve, s the arc-length, and $\mathbf{p}$, $\mathbf{b}$ the unit principal normal and binormal respectively. Then, as in Art. 56,

$$\operatorname{rot}\mathbf{n}=\kappa\mathbf{b}.$$

The condition (29) for a Lamé family may therefore be expressed

$$\operatorname{div} \frac{d}{ds}(\kappa \mathbf{b}) = 0,$$

or, in virtue of the Serret-Frenet formulae,

$$\operatorname{div}(\kappa' \mathbf{b} - \kappa\tau \mathbf{p}) = 0 \quad \text{.....................(32).}$$

Thus:

If a family of surfaces forms part of a triply orthogonal system, its orthogonal trajectories must be such that $\operatorname{div}(\kappa'\mathbf{b} - \kappa\tau\mathbf{p})$ *vanishes identically.*

The vector $\kappa'\mathbf{b} - \kappa\tau\mathbf{p}$ will itself vanish identically only if the trajectories are plane curves and κ' is zero. These curves then constitute a normal congruence of circles, commonly called a "cyclic congruence." Thus:

Any Lamé family possessing the property that $\mathbf{n} \cdot \nabla \operatorname{rot} \mathbf{n}$ *(but not* $\operatorname{rot} \mathbf{n}$*) vanishes identically, is such that its orthogonal trajectories constitute a cyclic congruence.*

More generally, if the orthogonal trajectories of a Lamé family are plane curves, these must satisfy the differential equation

$$\operatorname{div}(\kappa' \mathbf{b}) = 0,$$

that is

$$\operatorname{div}\left(\frac{\kappa'}{\kappa} \operatorname{rot} \mathbf{n}\right) = 0,$$

and therefore

$$(\operatorname{rot} \mathbf{n}) \cdot \nabla \frac{\kappa'}{\kappa} = 0.$$

Thus $\nabla\left(\dfrac{d}{ds}\log\kappa\right)$ is parallel to the osculating plane. And conversely:

If a normal congruence of plane curves is such that $\nabla\left(\dfrac{d}{ds}\log\kappa\right)$ *is everywhere parallel to the osculating plane, the surfaces orthogonal to these constitute a Lamé family.*

***62. The distance function for the family.** The distance function ψ for the Lamé family $w = \text{const.}$ is $\sqrt{c}$; for the length of the orthogonal trajectory intercepted between the adjacent surfaces w and $w + dw$ is $\sqrt{c}\,dw$. Since the unit normal is $\mathbf{r}_3/\sqrt{c}$, it follows,

from the values of the second derivatives of $\mathbf{r}$ previously found, that*

$$(\nabla\mathbf{n})\cdot\nabla\log c = \frac{1}{2\sqrt{c}}\left[\frac{a_3 c_1}{a^2 c}\mathbf{r}_1 + \frac{b_3 c_2}{b^2 c}\mathbf{r}_2 - \left(\frac{c_1^{\,2}}{ac^2} + \frac{c_2^{\,2}}{bc^2}\right)\mathbf{r}_3\right],$$

and therefore

$$\mathbf{n}\times\{(\nabla\mathbf{n})\cdot\nabla\log c\} = \frac{1}{2p}\left(\frac{a_3 c_1}{ac}\mathbf{r}_2 - \frac{b_3 c_2}{bc}\mathbf{r}_1\right).$$

Now, as in Art. 60, the divergence of this vector vanishes identically, in virtue of Lamé's relations between the derivatives of a, b, c. Hence the theorem:

For a Lamé family of surfaces the distance function ψ satisfies the differential equation

$$\operatorname{div}[\mathbf{n}\times(\nabla\mathbf{n})\cdot\nabla\log\psi] = 0 \quad\text{..............(33)}.$$

In this equation the operator ∇ is the three-parametric operator for space. We may establish a similar result in terms of the two-parametric operator ∇ for a surface $w = \text{const}$. The parameters on this surface are u, v, so that

$$\mathbf{n}\times(\nabla\sqrt{c}\cdot\nabla\mathbf{n}) = \mathbf{n}\times\left(\frac{a_3 c_1}{4a^2 c}\mathbf{r}_1 + \frac{b_3 c_2}{4b^2 c}\mathbf{r}_2\right)$$

$$= \frac{1}{4\sqrt{ab}}\left(\frac{a_3 c_1}{ac}\mathbf{r}_2 - \frac{b_3 c_2}{bc}\mathbf{r}_1\right),$$

and the two-parametric divergence of this vector vanishes, for the same reason as in the previous case. Thus, in terms of the operator $\overline{\nabla}$ of Art. 19, we may express this result:

On each surface of a Lamé family the distance function for the family satisfies the differential equation

$$\operatorname{Div}(\mathbf{n}\times\overline{\nabla}\psi) = 0 \quad\text{....................(34)}.$$

Expanding the first member, and remembering that the rotation of $\mathbf{n}$ relative to the surface is zero, we see that (34) is equivalent to

$$\mathbf{n}\cdot\operatorname{Rot}\overline{\nabla}\psi = 0 \quad\text{....................(35)}.$$

Then, since both $\overline{\nabla}\psi$ and its Rot are tangential to the surface, it

* The vector $(\nabla\mathbf{n})\cdot\mathbf{h}$ is to be interpreted as

$$\frac{1}{a}\mathbf{r}_1(\mathbf{n}_1\cdot\mathbf{h}) + \frac{1}{b}\mathbf{r}_2(\mathbf{n}_2\cdot\mathbf{h}) + \frac{1}{c}\mathbf{r}_3(\mathbf{n}_3\cdot\mathbf{h}).$$

Further explanation of this notation will be given in the next chapter.

follows that $\overline{\nabla}\psi$ is the gradient of some scalar function on the surface, or is zero. Thus*:

A necessary and sufficient condition that a family of surfaces may form part of a triply orthogonal system is that, on each surface, the vector $\overline{\nabla}\psi$ be the gradient of some scalar function.

In the case of a *family of parallel surfaces*, ψ is constant over each surface, and both $\nabla\psi$ and $\overline{\nabla}\psi$ vanish identically, and (34) is satisfied. For a *family of planes* the principal curvatures are zero, so that $\overline{\nabla}\psi$ again vanishes, and the necessary condition is satisfied. In the case of a *family of spheres* the principal curvatures are equal, and are constant for each surface. Then, when the components of $\nabla\psi$ in the principal directions are multiplied by the same constant, κ, the resultant vector, $\overline{\nabla}\psi$, is the gradient of $\kappa\psi$. Each of these families is therefore a Lamé family.

63. Surfaces of constant first curvature†. Consider next a family of surfaces, $w = \text{const.}$, for which the first curvature, J, is constant over each surface, but may vary from one surface to another. Then ∇J is parallel to $\mathbf{n}$, so that

$$\mathbf{n} \times \nabla \operatorname{div} \mathbf{n} = 0.$$

The equation (30) now takes the simple form

$$\operatorname{div}[(\operatorname{rot}\mathbf{n}) \cdot \nabla\mathbf{n}] = 0 \quad \text{..................(36).}$$

That is to say:

A family of surfaces of constant first curvature will be a Lamé family provided the unit normal satisfies (36).

For example, in the case of a *family of planes*, the derivative of $\mathbf{n}$ is zero for all directions perpendicular to $\mathbf{n}$. But $\operatorname{rot}\mathbf{n}$ is perpendicular to $\mathbf{n}$, by (6). Hence $(\operatorname{rot}\mathbf{n}) \cdot \nabla\mathbf{n}$ vanishes identically, and the condition (36) is satisfied.

Or again, for a *family of spheres*, if $\mathbf{a}$ is any vector perpendicular to $\mathbf{n}$,

$$\mathbf{a} \cdot \nabla\mathbf{n} = F\mathbf{a},$$

where F is a function of w only, being the curvature of a great circle of the sphere. In particular

$$(\operatorname{rot}\mathbf{n}) \cdot \nabla\mathbf{n} = F \operatorname{rot}\mathbf{n},$$

* *Liouville's Journal*, loc. cit.

† *Annals of Mathematics*, loc. cit., p. 307.

and therefore

$$\operatorname{div}\,[(\operatorname{rot}\mathbf{n})\cdot\nabla\mathbf{n}] = \nabla F\cdot\operatorname{rot}\mathbf{n} = 0,$$

since ∇F is parallel to $\mathbf{n}$, and therefore perpendicular to $\operatorname{rot}\mathbf{n}$. Thus the equation (36) is again satisfied. Hence:

A singly infinite family of planes or of spheres is a Lamé family.

64. Lamé family of developable surfaces. We shall conclude this discussion of Lamé families by proving the following theorem:

If one family of a triply orthogonal system is a family of developable surfaces, either it consists of planes, or else a second family is also one of developables and the third is one of parallels.

Suppose, for instance, that the surfaces $u=$ const. are developables, and that the principal curvature in the direction of $\mathbf{r}_2$ is zero. We have seen that the principal curvatures have the values*

$$-\frac{b_1}{2b\sqrt{a}} \quad\text{and}\quad -\frac{c_1}{2c\sqrt{a}}.$$

Thus $b_1=0$, and consequently, in virtue of the relation

$$\frac{\partial}{\partial w}\left(\frac{b_1}{b}\right) = \frac{b_1a_3}{2ba} + \frac{b_3c_1}{2bc} - \frac{b_1b_3}{2b^2},$$

either $c_1=0$ or $b_3=0$. In the former case both principal curvatures of a surface $u=$ const. are zero, and these surfaces are therefore planes. In the second case the surfaces $w=$ const. are developables, since one of the principal curvatures is $-b_3/2b\sqrt{c}$; and the surfaces $v=$ const. are parallels, because for this family $\operatorname{rot}\mathbf{n}$ then vanishes identically, being given by

$$\operatorname{rot}\mathbf{n} = \frac{-1}{2p\sqrt{b}}(b_3\mathbf{r}_1 - b_1\mathbf{r}_3).$$

Hence the above theorem.

* Vol. I, Art. 108.

EXAMPLES VI

1. Deduce theorem (1) of Art. 52 from the corresponding formula $J = -\operatorname{Div} \mathbf{n}$, using Cartesian coordinate axes, with the z-axis parallel to the normal at the point considered.

2. The first curvature of a surface $u = \text{const.}$ of an oblique system is given by

$$J = -\frac{1}{p}\left[\frac{\partial}{\partial u}\left(\frac{A}{\sqrt{A}}\right) + \frac{\partial}{\partial v}\left(\frac{H}{\sqrt{A}}\right) + \frac{\partial}{\partial w}\left(\frac{G}{\sqrt{A}}\right)\right].$$

Give similar expressions for the first curvatures of the other parametric surfaces through the point.

3. Deduce the relation $\mathbf{n} \cdot \operatorname{rot} \mathbf{n}$, satisfied by the unit normal, from formula (22) of Art. 43.

4. From the fact that, in the case of a family of parallel surfaces $w = \text{const.}$, $(\nabla w)^2$ is a function of w only, deduce that $\operatorname{rot} \mathbf{n}$ vanishes identically.

5. If $\mathbf{a}$, $\mathbf{b}$ are unit vectors, tangential to orthogonal systems of curves on the surfaces $w = \text{const.}$, and ∇ is the three-parametric operator for space, show that the second curvature of a surface $w = \text{const.}$ is given by

$$K = (\mathbf{a} \cdot \nabla \mathbf{a}) \cdot (\mathbf{b} \cdot \nabla \mathbf{b}) - (\mathbf{b} \cdot \nabla \mathbf{a}) \cdot (\mathbf{a} \cdot \nabla \mathbf{b}).$$

6. Show that the surfaces $w = \text{const.}$, of an oblique system, will be isometric provided

$$\frac{p}{C}\left[\frac{\partial}{\partial u}\left(\frac{G}{p}\right) + \frac{\partial}{\partial v}\left(\frac{F}{p}\right) + \frac{\partial}{\partial w}\left(\frac{C}{p}\right)\right]$$

is a function of w only.

7. From (24) of Art. 58 deduce that, if a family of parallel surfaces is isometric, the first curvature is constant over each surface of the family. Hence show that K also is constant over each surface.

8. If $\mathbf{n}$ is the unit normal for a family of surfaces, and Div denotes the divergence relative to one of the surfaces, prove that

$$\operatorname{Div} \operatorname{rot} \mathbf{n} = 0.$$

9. We may express the second curvature of a member of the family of surfaces, $\phi = \text{const.}$, in terms of two-parametric differential invariants of ψ and ρ relative to that surface. With the notation of Arts. 56 and 57, since $\mathbf{p}$, $\mathbf{b}$ are unit tangents to orthogonal curves on the surface, we have

$$-K = \operatorname{Div}(\mathbf{p} \operatorname{Div} \mathbf{p} + \mathbf{b} \operatorname{Div} \mathbf{b}).$$

But
$$\operatorname{Div} \mathbf{b} = \operatorname{Div}(\rho \operatorname{rot} \mathbf{n}) = (\operatorname{Grad} \rho) \cdot \operatorname{rot} \mathbf{n},$$

by Ex. 8; and this result may be expressed

$$\operatorname{Div} \mathbf{b} = (\operatorname{Grad} \rho) \cdot (\kappa \mathbf{b}) = \mathbf{b} \cdot \operatorname{Grad} \log \rho.$$

Similarly we have

$$\begin{aligned}\operatorname{Div}\mathbf{p} &= \operatorname{Div}(\rho\kappa\mathbf{p}) = \kappa\mathbf{p}\cdot\operatorname{Grad}\rho - \rho\operatorname{Div}\operatorname{Grad}\log\psi \\ &= \mathbf{p}\cdot\operatorname{Grad}\log\rho - \rho\nabla'^2\log\psi,\end{aligned}$$

where, for the moment, we distinguish the operator ∇^2 for the surface by a prime. Consequently

$$\mathbf{p}\operatorname{Div}\mathbf{p} + \mathbf{b}\operatorname{Div}\mathbf{b} = \operatorname{Grad}\log\rho + \theta\operatorname{Grad}\log\psi,$$

where $$\theta = \rho^2\nabla'^2\log\psi = \nabla'^2\log\psi/(\operatorname{Grad}\log\psi)^2.$$

Hence the result

$$K = \operatorname{Div}(\operatorname{Grad}\log\kappa - \theta\operatorname{Grad}\log\psi) \quad\text{.....................(i).}$$

The lines of equidistance on a surface $\phi=$const., and their orthogonal trajectories, will constitute an isometric system on that surface provided θ is a function of ψ only. If this relation holds and we write

$$F = \int\frac{\theta}{\psi}\,d\psi,$$

the above formula for K may be expressed

$$\begin{aligned}K &= \operatorname{Div}(\operatorname{Grad}\log\kappa - \operatorname{Grad}F) \\ &= \nabla'^2(\log\kappa - F) \quad\text{.......................................(ii).}\end{aligned}$$

10. Various alternative forms may be found for the condition that a given family of surfaces may be a Lamé family. For instance, from the identity

$$\operatorname{rot}(\mathbf{n}\times\operatorname{rot}\mathbf{n}) = (\operatorname{rot}\mathbf{n})\cdot\nabla\mathbf{n} - \mathbf{n}\cdot\nabla\operatorname{rot}\mathbf{n} - (\operatorname{div}\mathbf{n})\operatorname{rot}\mathbf{n},$$

on taking the divergence of both members, we see that (29) of Art. 60 is equivalent to

$$\operatorname{div}[(\operatorname{rot}\mathbf{n})\cdot\nabla\mathbf{n} - (\operatorname{div}\mathbf{n})\operatorname{rot}\mathbf{n}] = 0 \quad\text{.....................(i).}$$

Again, taking the gradient of both members of the identity

$$\mathbf{n}\cdot\operatorname{rot}\mathbf{n} = 0,$$

we obtain $$(\operatorname{rot}\mathbf{n})\cdot\nabla\mathbf{n} + \mathbf{n}\cdot\nabla\operatorname{rot}\mathbf{n} + \mathbf{n}\times\operatorname{rot}\operatorname{rot}\mathbf{n} = 0.$$

The condition (29) is therefore equivalent to

$$\operatorname{div}[(\operatorname{rot}\mathbf{n})\cdot\nabla\mathbf{n} + \mathbf{n}\times\operatorname{rot}\operatorname{rot}\mathbf{n}] = 0 \quad\text{..................(ii),}$$

or, in virtue of (30), to

$$\operatorname{div}[\mathbf{n}\times(\nabla\operatorname{div}\mathbf{n} + \operatorname{rot}\operatorname{rot}\mathbf{n})] = 0 \quad\text{..................(iii),}$$

and therefore to $$\operatorname{div}[\mathbf{n}\times(2\nabla\operatorname{div}\mathbf{n} - \nabla^2\mathbf{n})] = 0 \quad\text{..................(iv).}$$

11. Prove that, for a family of parallel surfaces, $(\nabla\mathbf{n})\cdot\nabla\log\psi$ vanishes identically, and that, for a family of planes, $\mathbf{n}\times(\nabla\mathbf{n})\cdot\nabla\log\psi$ is everywhere equal to zero. Hence these are Lamé families.

12. Prove that, for a family of surfaces, the *normal derivative of the first curvature* is given by

$$\mathbf{n}\cdot\nabla J = J^2 - 2K + \frac{1}{\psi}\nabla'^2\psi,$$

where ψ is the distance function for the family, and $\nabla'^2\psi$ is the two-parametric invariant Div Grad ψ for the surface of the family.

For $$\mathbf{n}\cdot\nabla J = -\mathbf{n}\cdot\nabla\,\mathrm{div}\,\mathbf{n}$$
$$= -\mathbf{n}\cdot(\mathrm{rot}\,\mathrm{rot}\,\mathbf{n}+\nabla^2\mathbf{n})$$
$$= J^2-2K+(\mathrm{rot}\,\mathbf{n})^2-\mathbf{n}\cdot\mathrm{rot}\,\mathrm{rot}\,\mathbf{n} \qquad \text{by (8)}$$
$$= J^2-2K+\mathrm{div}\,(\mathbf{n}\times\mathrm{rot}\,\mathbf{n}).$$

Now $\mathbf{n}\times\mathrm{rot}\,\mathbf{n} = \mathrm{Grad}\,\log\psi$, and therefore, by (24) of Art. 43,

$$\mathrm{div}\,(\mathbf{n}\times\mathrm{rot}\,\mathbf{n}) = \mathrm{div}\left(\frac{1}{\psi}\,\mathrm{Grad}\,\psi\right)$$
$$= \frac{1}{\psi}\,\mathrm{Div}\,\mathrm{Grad}\,\psi = \frac{1}{\psi}\,\nabla'^2\psi.$$

Hence the result.

13. Show that the *normal derivative of the second curvature*, K, for a family of surfaces is given by

$$\mathbf{n}\cdot\nabla K = JK + \frac{1}{\psi}\,\mathrm{Div}\,(K\nabla^*\psi),$$

where ψ has the usual meaning, and ∇^* is the operator defined in Art. 19.

14. Show that, if a family of parallel surfaces is isometric, the first and second curvatures are constant over each member of the family.

CHAPTER VII

TENSORS OF THE SECOND ORDER. DYADICS

65. Dyads and Dyadics. In order to secure greater freedom of analysis, we shall here introduce the reader to the more elementary properties of dyadics, which are substantially tensors of the second order. The elements of the subject will be sufficient for our purpose*.

Consider the vector $\mathbf{s}$ defined by the equation

$$\mathbf{s} = (\mathbf{r} \cdot \mathbf{a})\,\mathbf{l} + (\mathbf{r} \cdot \mathbf{b})\,\mathbf{m} + (\mathbf{r} \cdot \mathbf{c})\,\mathbf{n},$$

the scalar coefficient in any term of the second member being the scalar product of $\mathbf{r}$ and another vector. The equation may be more briefly written

$$\mathbf{s} = \mathbf{r} \cdot (\mathbf{a\,l} + \mathbf{b\,m} + \mathbf{c\,n}) \quad \text{.................(1)},$$

in which $(\mathbf{a\,l} + \mathbf{b\,m} + \mathbf{c\,n})$ is a *dyadic*, of which $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are the *antecedents* and $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ the *consequents*. Each term of the dyadic, such as $\mathbf{a\,l}$, is called a *dyad*; and, in the second member of (1), the scalar product of $\mathbf{r}$ and any antecedent is the coefficient of the consequent of the same dyad. In the above equation the dyadic follows the vector $\mathbf{r}$, and is therefore called a *postfactor*. But the relation may be equally well expressed

$$\mathbf{s} = (\mathbf{l\,a} + \mathbf{m\,b} + \mathbf{n\,c}) \cdot \mathbf{r} \quad \text{.................(2)},$$

in which the scalar product of $\mathbf{r}$ and any consequent is the coefficient of the antecedent of the same dyad. In (2) the dyadic occurs as a *prefactor* to $\mathbf{r}$. It is the *conjugate* of the dyadic in (1). Thus the conjugate of any dyadic is obtained from it by interchanging the antecedent and the consequent in each dyad. Any dyadic used as a prefactor in the above manner is equivalent to its conjugate used as a postfactor.

* The reader who desires a more complete treatment is referred to the author's *Advanced Vector Analysis*, Chapters V and VII, or E. B. Wilson's *Vector Analysis*, Chapter V.

Dyadics are conveniently denoted by Greek capitals, Φ, Ψ, Ω, etc. The conjugate of Φ is then represented by Φ_c. Thus, if Φ is the dyadic in (1), we have

$$\mathbf{s} = \mathbf{r} \cdot \Phi = \Phi_c \cdot \mathbf{r}.$$

It is clear from the above that the *distributive law* holds for the operation of a dyadic upon a sum of vectors. Thus

$$\Phi \cdot (\mathbf{r} + \mathbf{s}) = \Phi \cdot \mathbf{r} + \Phi \cdot \mathbf{s}.$$

Similarly, if we define the sum of two dyadics Φ, Ψ as the dyadic $\Phi + \Psi$ which is formed by adding the dyads of Φ to those of Ψ, it is obvious that

$$\mathbf{r} \cdot (\Phi + \Psi) = \mathbf{r} \cdot \Phi + \mathbf{r} \cdot \Psi.$$

The expression $\mathbf{r} \cdot \Phi$ is called the *direct product*, or *dot product*, of $\mathbf{r}$ with the dyadic Φ used as a postfactor.

66. Open products of vectors. The symbolic product represented by the juxtaposition of the two vectors in any dyad may be called the *open product* of the two vectors. It is neither a scalar quantity nor a vector, but is used as an operator as explained above. Such products do not obey the associative law. They do, however, obey the distributive law, as will be seen from the following.

Two dyadics, Φ and Ψ, are said to be equal if

$$\mathbf{r} \cdot \Phi = \mathbf{r} \cdot \Psi, \quad \text{or} \quad \Phi \cdot \mathbf{r} = \Psi \cdot \mathbf{r},$$

for all values of the vector $\mathbf{r}$. On this understanding, if one vector of a dyad be expressed as a sum of vectors, the open product may be expanded according to the distributive law; that is to say

$$\mathbf{a}(\mathbf{l} + \mathbf{m} + \mathbf{n} + \ldots) = \mathbf{a}\,\mathbf{l} + \mathbf{a}\,\mathbf{m} + \mathbf{a}\,\mathbf{n} + \ldots.$$

For these two dyadics, forming direct products with $\mathbf{r}$ in the same way, give the same result. Continued application of this principle shows that, if each vector of a dyad be expressed as the sum of any number of vectors, the open product may be expanded according to the distributive law of algebra, *provided the order of the factors in each term is maintained.* This stipulation is essential; for clearly the dyad $\mathbf{a}\,\mathbf{l}$ is not equal to $\mathbf{l}\,\mathbf{a}$.

Any dyadic may be reduced to the sum of three dyads, of which either the antecedents or the consequents may be arbitrarily chosen,

provided they are not coplanar. Suppose, for instance, that it is desired to express the dyadic as the sum of three dyads whose antecedents are **a**, **b**, **c**. First all the antecedents may be expressed in terms of these vectors, and each of the dyads then expanded as the sum of three others. Then all the new dyads, whose antecedents have the direction of **a**, may be combined, and their sum expressed as a single dyad $\mathbf{a\,u}$. Similarly the remainder of the dyadic may be arranged in the form $\mathbf{b\,v}+\mathbf{c\,w}$; so that the whole dyadic is expressible as

$$\mathbf{a\,u}+\mathbf{b\,v}+\mathbf{c\,w},$$

in accordance with the previous statement. Similarly it might be expressed as the sum of three dyads with arbitrarily chosen consequents, provided these are not coplanar. If all the antecedents, or all the consequents, of a dyadic are coplanar, it is expressible as the sum of two dyads. In this case the dyadic is said to be *planar*. If the antecedents and the consequents are all coplanar, the dyadic is said to be *uniplanar*. Similarly, if all the antecedents or all the consequents are parallel, the dyadic is expressible as a single dyad, and is said to be *linear*. If the antecedents and the consequents are all parallel, the dyadic is said to be *unilinear*. When, however, a dyadic is neither planar nor linear it is said to be *complete*. In this case it cannot be represented by the sum of less than three dyads.

67. Products of dyads and of dyadics. The *direct product* of the dyads $\mathbf{a\,l}$ and $\mathbf{p\,s}$, the former used as a prefactor and the latter as a postfactor, is denoted by $(\mathbf{a\,l})\cdot(\mathbf{p\,s})$, and is defined as being equal to $(\mathbf{l}\cdot\mathbf{p})(\mathbf{a\,s})$. Thus

$$(\mathbf{a\,l})\cdot(\mathbf{p\,s})=\mathbf{a}\,(\mathbf{l}\cdot\mathbf{p})\,\mathbf{s}=(\mathbf{l}\cdot\mathbf{p})\,(\mathbf{a\,s}).$$

The adjacent vectors **l**, **p** in the first member form a scalar product, which becomes a numerical coefficient to the dyad whose antecedent and consequent are the antecedent of the first and the consequent of the second respectively.

The direct product of two dyadics may be defined by the formal expansion, according to the distributive law, of two sums of dyads. Thus if

$$\Phi=\mathbf{a\,l}+\mathbf{b\,m}+\ldots,$$

and

$$\Psi=\mathbf{p\,s}+\mathbf{q\,t}\;+\ldots,$$

their direct product is given by

$$\Phi \cdot \Psi = (\mathbf{a\,l}) \cdot (\mathbf{p\,s}) + (\mathbf{a\,l}) \cdot (\mathbf{q\,t}) + \ldots$$
$$= \Sigma\,(\mathbf{l} \cdot \mathbf{p})(\mathbf{a\,s}).$$

And it is clear that the distributive law holds for such products of dyadics, so that

$$\Phi \cdot (\Psi + \Omega + \ldots) = \Phi \cdot \Psi + \Phi \cdot \Omega + \ldots,$$

and $$(\Psi + \Omega + \ldots) \cdot \Phi = \Psi \cdot \Phi + \Omega \cdot \Phi + \ldots.$$

The *associative law* also holds for the direct product of several dyadics; that is to say $(\Phi \cdot \Psi) \cdot \Omega = \Phi \cdot (\Psi \cdot \Omega)$. For, in each of these products, the consequents of the first dyadic form scalar products with the antecedents of the second, and the consequents of the second with the antecedents of the third. The antecedents of the first form open products with the consequents of the third, and the result is the same in both cases. The brackets are therefore unnecessary, and the product may be written simply $\Phi \cdot \Psi \cdot \Omega$. The same reasoning shows that the associative law holds for the direct product of any number of dyadics, *provided the order of the dyadics in the product is not altered.*

In the same way it follows that the associative law holds for direct products such as

$$\mathbf{r} \cdot \Phi \cdot \Psi \cdot \Omega, \quad \Phi \cdot \Psi \cdot \Omega \cdot \mathbf{r}, \quad \mathbf{r} \cdot \Phi \cdot \Psi \cdot \Omega \cdot \mathbf{s},$$

the vector factors occurring in the extreme positions. In the first of these expressions, $\mathbf{r}$ forms scalar products with the antecedents of Φ; and, in virtue of the distributive law, the result is the same in whatever manner the factors are bracketed, provided their order is not changed.

68. Nonion form. Scalar and vector of a dyadic. Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be three mutually perpendicular unit vectors forming a right-handed system. If all the antecedents and consequents of a dyadic, Φ, are expressed in terms of these vectors, and the open products then expanded according to the distributive law, only nine independent dyads will result. Combining similar dyads we may thus express the dyadic in the form

$$\Phi = \left.\begin{aligned} & a_1\mathbf{i\,i} + a_2\mathbf{i\,j} + a_3\mathbf{i\,k} \\ & + b_1\mathbf{j\,i} + b_2\mathbf{j\,j} + b_3\mathbf{j\,k} \\ & + c_1\mathbf{k\,i} + c_2\mathbf{k\,j} + c_3\mathbf{k\,k} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(3).$$

This is called the *nonion form* of Φ. Two dyadics are equal if their nonion forms are identical.

Let Φ be expressed in any form, say

$$\Phi = \mathbf{a}\,\mathbf{l} + \mathbf{b}\,\mathbf{m} + \mathbf{c}\,\mathbf{n} + \dots.$$

Then the *scalar* of Φ, denoted by Φ_s, may be defined by

$$\Phi_s = \mathbf{a}\cdot\mathbf{l} + \mathbf{b}\cdot\mathbf{m} + \mathbf{c}\cdot\mathbf{n} + \dots \quad \text{..............(4).}$$

It is thus equal to the sum of the scalar products of the antecedent and consequent in each term. And, since the scalar product of two sums of vectors may be expanded according to the distributive law, it follows that the value of Φ_s is independent of the form in which Φ is expressed. In terms of the coefficients of its nonion form

$$\Phi_s = a_1 + b_2 + c_3 \quad \text{.......................(5).}$$

Hence, whatever three mutually perpendicular vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are chosen,

$$\Phi_s = \mathbf{i}\cdot\Phi\cdot\mathbf{i} + \mathbf{j}\cdot\Phi\cdot\mathbf{j} + \mathbf{k}\cdot\Phi\cdot\mathbf{k} \quad \text{...........(6).}$$

Also it is obvious that the scalar of a dyadic is equal to that of its conjugate.

Similarly the *vector* of Φ, which will be denoted by Φ_v, may be defined by

$$\Phi_v = \mathbf{a}\times\mathbf{l} + \mathbf{b}\times\mathbf{m} + \mathbf{c}\times\mathbf{n} + \dots \quad \text{...........(7).}$$

It is thus equal to the sum of the vector products of the antecedent and consequent in each dyad. And, as in the case of Φ_s, the value of the vector of a dyadic is independent of the form in which the dyadic is expressed. If the nonion form is chosen, it follows from (3) that

$$\Phi_v = (b_3 - c_2)\,\mathbf{i} + (c_1 - a_3)\,\mathbf{j} + (a_2 - b_1)\,\mathbf{k} \quad \text{.........(8).}$$

Hence, for any three mutually perpendicular unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$,

$$\left.\begin{aligned} -\Phi_v &= \mathbf{i}\cdot\Phi\times\mathbf{i} + \mathbf{j}\cdot\Phi\times\mathbf{j} + \mathbf{k}\cdot\Phi\times\mathbf{k} \\ &= \mathbf{i}\times\Phi\cdot\mathbf{i} + \mathbf{j}\times\Phi\cdot\mathbf{j} + \mathbf{k}\times\Phi\cdot\mathbf{k} \end{aligned}\right\} \quad \text{........(9).}$$

Also from (7) it is clear that the vector of a dyadic is minus the vector of its conjugate.

69. The unit dyadic. Reciprocal dyadics. A unit dyadic, I, is defined by the properties that, for all values of the vector $\mathbf{r}$,

$$\mathbf{r}\cdot\mathrm{I} = \mathbf{r}, \quad \text{and} \quad \mathrm{I}\cdot\mathbf{r} = \mathbf{r}.$$

From these relations it is clear that the nonion form of the unit dyadic is

$$\mathrm{I} = \mathbf{i}\,\mathbf{i} + \mathbf{j}\,\mathbf{j} + \mathbf{k}\,\mathbf{k} \quad \text{.....................(10).}$$

Thus all unit dyadics are equal, since they have the same nonion form.

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be any three non-coplanar vectors, and $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ the reciprocal set, defined by

$$\mathbf{a}' = \mathbf{b} \times \mathbf{c}/p, \quad \mathbf{b}' = \mathbf{c} \times \mathbf{a}/p, \quad \mathbf{c}' = \mathbf{a} \times \mathbf{b}/p,$$

where $$p = [\mathbf{a}, \mathbf{b}, \mathbf{c}].$$

Then clearly and
$$\left.\begin{aligned} \mathbf{a} \cdot \mathbf{a}' &= \mathbf{b} \cdot \mathbf{b}' = \mathbf{c} \cdot \mathbf{c}' = 1 \\ \mathbf{a} \cdot \mathbf{b}' &= \mathbf{a} \cdot \mathbf{c}' = \text{etc.} \;\; = 0 \end{aligned}\right\} \quad \text{...............(11).}$$

In terms of these vectors the unit dyadic may be expressed

$$\left.\begin{aligned} \mathrm{I} &= \mathbf{a}\,\mathbf{a}' + \mathbf{b}\,\mathbf{b}' + \mathbf{c}\,\mathbf{c}' \\ &= \mathbf{a}'\mathbf{a} + \mathbf{b}'\mathbf{b} + \mathbf{c}'\mathbf{c} \end{aligned}\right\} \quad \text{..................(12).}$$

For the direct product of $\mathbf{r}$ with the second of these has the value*

$$(\mathbf{r} \cdot \mathbf{a}')\,\mathbf{a} + (\mathbf{r} \cdot \mathbf{b}')\,\mathbf{b} + (\mathbf{r} \cdot \mathbf{c}')\,\mathbf{c} = \mathbf{r},$$

and similarly $$(\mathbf{a}'\mathbf{a} + \mathbf{b}'\mathbf{b} + \mathbf{c}'\mathbf{c}) \cdot \mathbf{r} = \mathbf{r}.$$

The system $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ is its own reciprocal. Hence (10) is a particular case of (12).

The direct product of any dyadic Φ and the unit dyadic is equal to Φ. For, whatever be the value of $\mathbf{r}$,

$$(\Phi \cdot \mathrm{I}) \cdot \mathbf{r} = \Phi \cdot (\mathrm{I} \cdot \mathbf{r}) = \Phi \cdot \mathbf{r},$$

so that $$\Phi \cdot \mathrm{I} = \Phi,$$

and similarly $$\mathrm{I} \cdot \Phi = \Phi.$$

We may notice in passing that the uniplanar dyadic $\mathbf{i}\,\mathbf{i} + \mathbf{j}\,\mathbf{j}$ behaves as a unit dyadic for all vectors in the plane of $\mathbf{i}$ and $\mathbf{j}$. Such a dyadic is met with in geometry of a surface. For, if $\mathbf{a}$, $\mathbf{b}$ are unit tangents to the curves of an orthogonal system on the surface, the dyadic $\mathbf{a}\,\mathbf{a} + \mathbf{b}\,\mathbf{b}$ in direct multiplication with any surface vector leaves that vector unchanged; but, in direct multiplication with a normal vector, gives zero as the result. Thus $\mathbf{a}\,\mathbf{a} + \mathbf{b}\,\mathbf{b}$ acts as a unit dyadic for surface vectors, but as an annihilator for normal vectors.

* Cf. the author's *Elementary Vector Analysis*, Arts. 46, 47.

When two dyadics are so related that their direct product is equal to the unit dyadic, they are said to be *reciprocals*; or, each is the reciprocal of the other. The reciprocal of Φ is generally denoted by Φ^{-1}. Thus

$$\Phi \cdot \Phi^{-1} = \Phi^{-1} \cdot \Phi = \mathrm{I}.$$

In particular, if $$\Phi = a\,\mathbf{i}\,\mathbf{i} + b\,\mathbf{j}\,\mathbf{j} + c\,\mathbf{k}\,\mathbf{k},$$

then $$\Phi^{-1} = \frac{1}{a}\,\mathbf{i}\,\mathbf{i} + \frac{1}{b}\,\mathbf{j}\,\mathbf{j} + \frac{1}{c}\,\mathbf{k}\,\mathbf{k},$$

as is easily verified by forming their direct product. More generally, if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ are two sets of non-coplanar vectors, having for reciprocal sets $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ and $\mathbf{l}'$, $\mathbf{m}'$, $\mathbf{n}'$ respectively, then the reciprocal of the dyadic

$$\Phi = \mathbf{a}\,\mathbf{l} + \mathbf{b}\,\mathbf{m} + \mathbf{c}\,\mathbf{n}$$

is given by $$\Phi^{-1} = \mathbf{l}'\,\mathbf{a}' + \mathbf{m}'\,\mathbf{b}' + \mathbf{n}'\,\mathbf{c}' \quad \text{..............(13)},$$

in virtue of the relations (11).

The reciprocal of the direct product of any number of dyadics is equal to the product of their reciprocals taken in the reverse order. For, considering the case of two dyadics, we have

$$(\Phi \cdot \Psi) \cdot (\Psi^{-1} \cdot \Phi^{-1}) = \Phi \cdot \mathrm{I} \cdot \Phi^{-1} = \Phi \cdot \Phi^{-1} = \mathrm{I}.$$

Similarly the statement may be proved for the reciprocal of the product of any number of dyadics.

70. Symmetric and anti-symmetric dyadics. Another result, very similar to that just obtained, is that *the conjugate of the direct product of any number of dyadics is equal to the product of their conjugates taken in the reverse order.* For

$$(\Phi \cdot \Psi)_c \cdot \mathbf{r} = \mathbf{r} \cdot \Phi \cdot \Psi = (\Phi_c \cdot \mathbf{r}) \cdot \Psi = \Psi_c \cdot \Phi_c \cdot \mathbf{r},$$

and therefore $$(\Phi \cdot \Psi)_c = \Psi_c \cdot \Phi_c \quad \text{....................(14)}.$$

The proof may be extended to any number of dyadics.

If a dyadic is equal to its conjugate, it is said to be *symmetric* or *self-conjugate*. A dyadic which is the negative of its conjugate is said to be *anti-symmetric* or *anti-self-conjugate*. Thus, if Φ is symmetric,

$$\mathbf{r} \cdot \Phi = \Phi \cdot \mathbf{r}, \quad \Phi = \Phi_c,$$

while, for an anti-symmetric dyadic, Ω,

$$\mathbf{r} \cdot \Omega = -\Omega \cdot \mathbf{r}, \quad \Omega = -\Omega_c.$$

Any dyadic may be expressed as the sum of two parts, one of which is symmetric and the other anti-symmetric. For

$$\Phi = \tfrac{1}{2}(\Phi + \Phi_c) + \tfrac{1}{2}(\Phi - \Phi_c).$$

The first part is clearly symmetric, and the second anti-symmetric. We may also notice that:

The vector of a symmetric dyadic is zero. And conversely, if its vector is zero, the dyadic is symmetric.

This theorem may be proved by considering the nonion form of the dyadic. If the dyadic is symmetric it follows from (3) that

$$a_2 = b_1, \quad a_3 = c_1, \quad b_3 = c_2.$$

Consequently the vector Φ_v, as given by (8), is equal to zero. Conversely, if $\Phi_v = 0$, the above relations must be satisfied, and the dyadic is symmetric.

Any symmetric dyadic may be expressed in the form*

$$a\,\mathbf{i}\,\mathbf{i} + b\,\mathbf{j}\,\mathbf{j} + c\,\mathbf{k}\,\mathbf{k},$$

by suitably choosing the orthogonal system $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

71. Cross product of a dyadic and a vector. The *cross product*, or *skew product*, of a vector $\mathbf{r}$ with a dyad $\mathbf{a}\,\mathbf{l}$ used as a postfactor is defined by

$$\mathbf{r} \times (\mathbf{a}\,\mathbf{l}) = (\mathbf{r} \times \mathbf{a})\,\mathbf{l},$$

and is therefore also a dyad. The dyad may also occur as a prefactor. Thus

$$(\mathbf{a}\,\mathbf{l}) \times \mathbf{r} = \mathbf{a}\,(\mathbf{l} \times \mathbf{r}).$$

Similarly, in the case of a dyadic, we have

$$\mathbf{r} \times (\mathbf{a}\,\mathbf{l} + \mathbf{b}\,\mathbf{m} + \ldots) = (\mathbf{r} \times \mathbf{a})\,\mathbf{l} + (\mathbf{r} \times \mathbf{b})\,\mathbf{m} + \ldots,$$

the skew product being itself a dyadic. As in the case of a direct product, the associative law holds for the skew product of a vector with a direct product of dyadics, provided the vector is an extreme factor in the product. Thus

$$\Phi \cdot (\Psi \times \mathbf{r}) = (\Phi \cdot \Psi) \times \mathbf{r},$$

and so on, whatever the number of dyadics in the product. Moreover, since the dot and the cross may be interchanged in a scalar product of three vectors, it follows that

$$(\mathbf{r} \times \mathbf{s}) \cdot \Phi = \mathbf{r} \cdot (\mathbf{s} \times \Phi) = -\,\mathbf{s} \cdot (\mathbf{r} \times \Phi),$$

* Cf. the author's *Advanced Vector Analysis*, Art. 64.

and similarly

$$\Phi \cdot (\mathbf{r} \times \mathbf{s}) = (\Phi \times \mathbf{r}) \cdot \mathbf{s} = -(\Phi \times \mathbf{s}) \cdot \mathbf{r}.$$

But $\Phi \cdot (\mathbf{r} \times \mathbf{s})$ is not equal to $(\Phi \cdot \mathbf{r}) \times \mathbf{s}$. For, in the former, $\mathbf{s}$ enters into scalar triple products with the consequents of Φ; while, in the latter, it forms vector products with the antecedents of Φ.

We may notice in passing that any vector $\mathbf{s}$, used in cross multiplication with $\mathbf{r}$, may be replaced by the dyadic $\mathrm{I} \times \mathbf{s}$ or $\mathbf{s} \times \mathrm{I}$ used in direct multiplication. For

$$\mathbf{s} \times \mathbf{r} = (\mathrm{I} \cdot \mathbf{s}) \times \mathbf{r} = (\mathrm{I} \times \mathbf{s}) \cdot \mathbf{r},$$

since the dot and the cross may be interchanged in each scalar triple product. That the dyadics $\mathrm{I} \times \mathbf{s}$ and $\mathbf{s} \times \mathrm{I}$ are equal and anti-symmetric follows immediately from the expressions for $\mathbf{s}$ and I in terms of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. For if

$$\mathbf{s} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k},$$

it follows from (10) that each of the dyadics $\mathrm{I} \times \mathbf{s}$ and $\mathbf{s} \times \mathrm{I}$ is equal to the anti-symmetric dyadic

$$a\,(\mathbf{k}\,\mathbf{j} - \mathbf{j}\,\mathbf{k}) + b\,(\mathbf{i}\,\mathbf{k} - \mathbf{k}\,\mathbf{i}) + c\,(\mathbf{j}\,\mathbf{i} - \mathbf{i}\,\mathbf{j}).$$

72. Double multiplication of dyads and dyadics. The *double dot* product of two dyads is the scalar product of their antecedents multiplied by the scalar product of their consequents. This product is denoted by placing a double dot between the dyads. Thus

$$(\mathbf{a}\,\mathbf{l}) : (\mathbf{p}\,\mathbf{u}) = (\mathbf{a} \cdot \mathbf{p})\,(\mathbf{l} \cdot \mathbf{u}).$$

Clearly the order of the dyads may be reversed without altering the value of the product.

The double dot product of two dyadics is the sum of the double dot products of each dyad of the first with each dyad of the second; that is to say, it is the sum of the double products of dyads obtained by formal expansion according to the distributive law. Thus

$$\begin{aligned}(\mathbf{a}\,\mathbf{l} + \mathbf{b}\,\mathbf{m} + \ldots) &: (\mathbf{p}\,\mathbf{u} + \mathbf{q}\,\mathbf{v} + \ldots) \\ &= (\mathbf{a} \cdot \mathbf{p})\,(\mathbf{l} \cdot \mathbf{u}) + (\mathbf{a} \cdot \mathbf{q})\,(\mathbf{l} \cdot \mathbf{v}) + \ldots \\ &\quad + (\mathbf{b} \cdot \mathbf{p})\,(\mathbf{m} \cdot \mathbf{u}) + \ldots.\end{aligned}$$

Since scalar products obey the distributive law, the value of the above product is independent of the forms in which the two dyadics are expressed.

The *double cross* product of two dyads is the dyad whose antecedent is the vector product of their antecedents, and whose consequent is the vector product of their consequents. It is indicated by a double cross between the dyads. Thus

$$(\mathbf{a}\,\mathbf{l})\overset{\times}{\times}(\mathbf{p}\,\mathbf{u}) = (\mathbf{a}\times\mathbf{p})\,(\mathbf{l}\times\mathbf{u}).$$

The order of the dyads may be reversed without altering the value of this product.

The double cross product of two dyadics is the sum of the double cross products of each dyad of the first with each dyad of the second. And, as in the former case, this product is independent of the forms in which the two dyadics are expressed.

If we take the double cross product of two dyads, and form its double dot product with a third dyad, we obtain a scalar product of three dyads. Thus

$$\begin{aligned}(\mathbf{a}\,\mathbf{l})\overset{\times}{\times}(\mathbf{b}\,\mathbf{m}):(\mathbf{c}\,\mathbf{n}) &= (\mathbf{a}\times\mathbf{b}\;\;\mathbf{l}\times\mathbf{m}):(\mathbf{c}\,\mathbf{n})\\ &= [\mathbf{a},\,\mathbf{b},\,\mathbf{c}]\,[\mathbf{l},\,\mathbf{m},\,\mathbf{n}],\end{aligned}$$

being equal to the scalar triple product of the antecedents multiplied by the scalar triple product of the consequents. The result is clearly independent of the order of the dyads in the product. Similarly the triple product, $\Phi\overset{\times}{\times}\Psi:\Omega$, of three dyadics may be formed, by expanding the triple product of three sums of dyads according to the distributive law.

73. Second and third of a dyadic. The *second* of a dyadic Φ is defined as half the double cross product $\Phi\overset{\times}{\times}\Phi$. It is denoted by Φ_2. Thus

$$\Phi_2 = \tfrac{1}{2}\,\Phi\overset{\times}{\times}\Phi.$$

We have seen that Φ may be expressed as the sum of three dyads. If then

$$\Phi = \mathbf{a}\,\mathbf{l} + \mathbf{b}\,\mathbf{m} + \mathbf{c}\,\mathbf{n} \quad\text{..................(15)},$$

it is clear that

$$\Phi_2 = \mathbf{b}\times\mathbf{c}\;\;\mathbf{m}\times\mathbf{n} + \mathbf{c}\times\mathbf{a}\;\;\mathbf{n}\times\mathbf{l} + \mathbf{a}\times\mathbf{b}\;\;\mathbf{l}\times\mathbf{m}.$$

If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are non-coplanar, and likewise $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$, we may write this

$$\Phi_2 = [\mathbf{a},\,\mathbf{b},\,\mathbf{c}]\,[\mathbf{l},\,\mathbf{m},\,\mathbf{n}]\,(\mathbf{a}'\,\mathbf{l}' + \mathbf{b}'\,\mathbf{m}' + \mathbf{c}'\,\mathbf{n}') \quad\text{......(16)},$$

where $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ and $\mathbf{l}'$, $\mathbf{m}'$, $\mathbf{n}'$ as usual denote the reciprocal systems of vectors to $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ respectively.

The *third*, or *determinant*, of a dyadic Φ is defined as one-third of the double dot product of Φ with Φ_2. It is denoted by Φ_3. Thus

$$\Phi_3 = \tfrac{1}{3}\Phi : \Phi_2 = \tfrac{1}{6}\Phi : (\Phi \overset{\times}{\times} \Phi).$$

Using the values of Φ and Φ_2 given in (15) and (16) we have immediately, in virtue of (11),

$$\Phi_3 = [\mathbf{a},\ \mathbf{b},\ \mathbf{c}]\,[\mathbf{l},\ \mathbf{m},\ \mathbf{n}] \quad\text{..................(17).}$$

Forming the direct product of Φ_2, as given in (16), and Φ_c, as found from (15), we have

$$\Phi_2 \cdot \Phi_c = [\mathbf{a}, \mathbf{b}, \mathbf{c}]\,[\mathbf{l}, \mathbf{m}, \mathbf{n}]\,(\mathbf{a}'\,\mathbf{a} + \mathbf{b}'\,\mathbf{b} + \mathbf{c}'\,\mathbf{c}) = \Phi_3 \mathrm{I} \quad\text{...(18).}$$

Thus *the direct product of the second and the conjugate of a dyadic is equal to the product of the third and the unit dyadic.*

74. Scalar invariants of a dyadic. There are three scalar invariants of a dyadic which deserve mention. We may consider these in connection with the nonion form of the dyadic. With the notation of Art. 68 let

$$\Phi = \left.\begin{array}{l} \quad a_1\mathbf{i}\ \mathbf{i} + a_2\mathbf{i}\ \mathbf{j} + a_3\mathbf{i}\ \mathbf{k} \\ + b_1\mathbf{j}\ \mathbf{i} + b_2\mathbf{j}\ \mathbf{j} + b_3\mathbf{j}\ \mathbf{k} \\ + c_1\,\mathbf{k}\mathbf{i} + c_2\,\mathbf{k}\mathbf{j} + c_3\,\mathbf{k}\mathbf{k} \end{array}\right\} \quad\text{...........(19).}$$

The simplest of the three invariants is the *scalar* of Φ, already considered. In terms of the coefficients in (19) we have

$$\Phi_s = a_1 + b_2 + c_3 \quad\text{....................(20).}$$

The second invariant is the scalar of Φ_2. If we use the nonion form (19) to calculate the product $\tfrac{1}{2}\Phi \overset{\times}{\times} \Phi$, we find for the coefficient of $\mathbf{i}\,\mathbf{j}$ the value $b_3c_1 - b_1c_3$. But this is the cofactor of a_2 in the determinant

$$D = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad\text{....................(21).}$$

Denoting the cofactor of any element by the corresponding capital, we find in this way, for the second of Φ,

$$\Phi_2 = \left.\begin{array}{l} \quad A_1\mathbf{i}\ \mathbf{i} + A_2\mathbf{i}\ \mathbf{j} + A_3\mathbf{i}\ \mathbf{k} \\ + B_1\mathbf{j}\ \mathbf{i} + B_2\mathbf{j}\ \mathbf{j} + B_3\mathbf{j}\ \mathbf{k} \\ + C_1\,\mathbf{k}\mathbf{i} + C_2\,\mathbf{k}\mathbf{j} + C_3\,\mathbf{k}\mathbf{k} \end{array}\right\} \quad\text{...........(22).}$$

Thus the scalar of Φ_2 is given by

$$(\Phi_2)_s = A_1 + B_2 + C_3 \quad \text{..................(23)},$$

and this is the second scalar invariant.

The third invariant is the third of Φ, denoted by Φ_3. From (19) and (22), and the definition of Φ_3, it follows that

$$3\Phi_3 = \Phi : \Phi_2 = (a_1 A_1 + a_2 A_2 + a_3 A_3) \\ + (b_1 B_1 + b_2 B_2 + b_3 B_3) + (c_1 C_1 + c_2 C_2 + c_3 C_3) = 3D.$$

Consequently $$\Phi_3 = D \quad \text{..........................(24)},$$

or the third of Φ is equal to the determinant, D, of the coefficients in the nonion form of the dyadic. Hence the name "determinant" of Φ, sometimes used for this invariant.

We may remark that the determinant of Φ_2 is the determinant of cofactors of the elements of D, and is therefore equal to D^2. Thus the determinant of the second of Φ is equal to the square of the determinant of Φ.

Lastly, the nonion form of the reciprocal of Φ may be readily deduced from (18). For, since

$$\Phi_2 \cdot \Phi_c = DI,$$

we have, on forming the direct product of each member with Φ_c^{-1},

$$D\Phi_c^{-1} = \Phi_2 \cdot \Phi_c \cdot \Phi_c^{-1} = \Phi_2.$$

Consequently

$$\begin{aligned} D\Phi^{-1} &= (\Phi_2)_c \\ &= \left. \begin{aligned} &A_1\mathbf{i}\,\mathbf{i} + B_1\mathbf{i}\,\mathbf{j} + C_1\mathbf{i}\,\mathbf{k} \\ + &A_2\mathbf{j}\,\mathbf{i} + B_2\mathbf{j}\,\mathbf{j} + C_2\mathbf{j}\,\mathbf{k} \\ + &A_3\mathbf{k}\,\mathbf{i} + B_3\mathbf{k}\,\mathbf{j} + C_3\mathbf{k}\,\mathbf{k} \end{aligned} \right\} \text{............(25)}. \end{aligned}$$

Dyadics formed with the Operator ∇

75. The dyadics $\nabla\mathbf{s}$ and $\mathbf{s}\nabla$. We have already seen that the operator ∇ may be applied to a vector function $\mathbf{s}$ so as to produce the divergence of $\mathbf{s}$, denoted by $\nabla \cdot \mathbf{s}$, and the rotation of $\mathbf{s}$, denoted by $\nabla \times \mathbf{s}$; while $\mathbf{c} \cdot \nabla\mathbf{s}$ has been interpreted as the result of the operation of $\mathbf{c} \cdot \nabla$ upon $\mathbf{s}$. Hitherto we have attached no meaning to $\nabla\mathbf{s}$. We agree to interpret this by analogy with $\nabla\phi$. Adopting

the notation of Art. 42 for oblique curvilinear coordinates, so that the operator ∇ *for space* is given by

$$\nabla = \mathbf{l}\frac{\partial}{\partial u} + \mathbf{m}\frac{\partial}{\partial v} + \mathbf{n}\frac{\partial}{\partial w} \quad \text{.................(26)},$$

we define $\nabla\mathbf{s}$ as the dyadic

$$\nabla\mathbf{s} = \mathbf{l}\frac{\partial\mathbf{s}}{\partial u} + \mathbf{m}\frac{\partial\mathbf{s}}{\partial v} + \mathbf{n}\frac{\partial\mathbf{s}}{\partial w} \quad \text{..............(27)},$$

and we agree to interpret $\mathbf{s}\nabla$ as the conjugate dyadic

$$\mathbf{s}\nabla = \frac{\partial\mathbf{s}}{\partial u}\mathbf{l} + \frac{\partial\mathbf{s}}{\partial v}\mathbf{m} + \frac{\partial\mathbf{s}}{\partial w}\mathbf{n} \quad \text{.................(28)}.$$

The former of these dyadics occurs frequently; but the latter will be introduced only occasionally. It is clear that the scalar of the dyadic $\nabla\mathbf{s}$ is equal to $\nabla\cdot\mathbf{s}$, and its vector to $\nabla\times\mathbf{s}$. Similarly the scalar and the vector of $\mathbf{s}\nabla$ are $\nabla\cdot\mathbf{s}$ and $-\nabla\times\mathbf{s}$ respectively.

The vector $\mathbf{c}\cdot\nabla\mathbf{s}$ may now be regarded either as the direct product of $\mathbf{c}$ with $\nabla\mathbf{s}$ used as a postfactor, or as the result of operating on $\mathbf{s}$ with $\mathbf{c}\cdot\nabla$. From either point of view its value is

$$(\mathbf{c}\cdot\mathbf{l})\frac{\partial\mathbf{s}}{\partial u} + (\mathbf{c}\cdot\mathbf{m})\frac{\partial\mathbf{s}}{\partial v} + (\mathbf{c}\cdot\mathbf{n})\frac{\partial\mathbf{s}}{\partial w} \quad \text{...........(29)}.$$

Similarly, if $\mathbf{s}$ and $\mathbf{s}+d\mathbf{s}$ are the values of the function $\mathbf{s}$ at the adjacent points $\mathbf{r}$ and $\mathbf{r}+d\mathbf{r}$ respectively, since

$$d\mathbf{r} = \mathbf{r}_1\,du + \mathbf{r}_2\,dv + \mathbf{r}_3\,dw,$$

we have

$$\begin{aligned} d\mathbf{r}\cdot\nabla\mathbf{s} &= (\mathbf{r}_1\,du + \mathbf{r}_2\,dv + \ldots)\cdot\left(\mathbf{l}\frac{\partial\mathbf{s}}{\partial u} + \mathbf{m}\frac{\partial\mathbf{s}}{\partial v} + \ldots\right) \\ &= \frac{\partial\mathbf{s}}{\partial u}du + \frac{\partial\mathbf{s}}{\partial v}dv + \frac{\partial\mathbf{s}}{\partial w}dw \\ &= d\mathbf{s} \quad \text{...(30)}, \end{aligned}$$

as in the case of a scalar function.

If, in the above, we take $\mathbf{s}$ as the function $\nabla\phi$, we obtain the dyadic $\nabla\nabla\phi$, whose scalar is $\nabla\cdot\nabla\phi$ or $\nabla^2\phi$, and whose vector is $\nabla\times\nabla\phi$ or rot $\nabla\phi$, which vanishes identically. Since the vector of $\nabla\nabla\phi$ is zero at all points, this dyadic is *symmetric* (Art. 70).

76. Formulae of expansion. The derivative of the open product represented by the dyad $\mathbf{s}\,\mathbf{t}$ is defined by the ordinary rule

for differentiating a product. Thus, if u is the variable of differentiation,

$$\frac{\partial}{\partial u}(\mathbf{s}\,\mathbf{t}) = \frac{\partial \mathbf{s}}{\partial u}\mathbf{t} + \mathbf{s}\frac{\partial \mathbf{t}}{\partial u},$$

the result being a dyadic. Similarly the derivative of a dyadic is defined as the sum of the derivatives of its dyads.

The operator ∇ may be applied to a dyadic Φ to produce functions represented by $\nabla \cdot \Phi$ and $\nabla \times \Phi$. As in the case of the divergence and rotation of a vector, these are defined by

$$\left.\begin{aligned} \nabla \cdot \Phi &= \mathbf{l} \cdot \frac{\partial \Phi}{\partial u} + \mathbf{m} \cdot \frac{\partial \Phi}{\partial v} + \mathbf{n} \cdot \frac{\partial \Phi}{\partial w} \\ \nabla \times \Phi &= \mathbf{l} \times \frac{\partial \Phi}{\partial u} + \mathbf{m} \times \frac{\partial \Phi}{\partial v} + \mathbf{n} \times \frac{\partial \Phi}{\partial w} \end{aligned}\right\} \quad \text{......(31)},$$

the former being a vector and the latter a dyadic. Then, corresponding to the results of Chapter V, we have the formulae

$$\left.\begin{aligned} \nabla \times \nabla \mathbf{s} &= 0 \\ \nabla \cdot \nabla \times \Phi &= 0 \\ \nabla \cdot \nabla \mathbf{s} &= \nabla^2 \mathbf{s} \end{aligned}\right\} \quad \text{....................(32)},$$

∇ being the three-parametric operator for space. These formulae may be easily verified, by employing the expression for ∇ corresponding to fixed rectangular axes, viz.

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

In Art. 44 an expansion formula was given for the gradient of a scalar product $\mathbf{u} \cdot \mathbf{v}$. Sometimes an alternative formula is more convenient; and with this we may bracket a similar one. These are

$$\left.\begin{aligned} \nabla(\mathbf{u} \cdot \mathbf{v}) &= \nabla \mathbf{u} \cdot \mathbf{v} + \nabla \mathbf{v} \cdot \mathbf{u} \\ \nabla(\mathbf{u} \times \mathbf{v}) &= \nabla \mathbf{u} \times \mathbf{v} - \nabla \mathbf{v} \times \mathbf{u} \end{aligned}\right\} \quad \text{............(33)},$$

in which the operator ∇ applies only to the vector adjacent to it, unless brackets are used to indicate the contrary. Thus $\nabla \mathbf{u} \cdot \mathbf{v}$ is the direct product of $\mathbf{v}$ with the dyadic $\nabla \mathbf{u}$ used as a prefactor; and so on. These formulae follow immediately on using the value of ∇ expressed in (26).

Finally, if ϕ and $\mathbf{u}$ are a scalar and a vector function, and Φ a variable dyadic, the formulae

$$\left.\begin{aligned}\nabla(\phi\mathbf{u}) &= \nabla\phi\mathbf{u} + \phi\nabla\mathbf{u}\\ \nabla\cdot(\phi\Phi) &= \nabla\phi\cdot\Phi + \phi\nabla\cdot\Phi\\ \nabla\times(\phi\Phi) &= \nabla\phi\times\Phi + \phi\nabla\times\Phi\end{aligned}\right\} \quad \text{.............(34)}$$

correspond to (18), (25) and (26) of Chapter V, and may be proved in a similar manner.

77. The operator ∇ for a given surface. The operator ∇ considered in the two preceding articles is the three-parametric operator for space. Most of the properties and formulae established are true also when ∇ is the two-parametric operator for a given surface. In this case, if the coordinate curves are orthogonal, we have

$$\nabla = \frac{1}{E}\mathbf{r}_1\frac{\partial}{\partial u} + \frac{1}{G}\mathbf{r}_2\frac{\partial}{\partial v},$$

so that

$$\nabla\mathbf{s} = \frac{1}{E}\mathbf{r}_1\mathbf{s}_1 + \frac{1}{G}\mathbf{r}_2\mathbf{s}_2 \quad \text{.................(35)},$$

and $\mathbf{s}\nabla$ is the conjugate of this dyadic. Also, as we have already seen,

$$d\mathbf{s} = d\mathbf{r}\cdot\nabla\mathbf{s} \quad \text{.......................(36)}.$$

The formulae of expansion, (33) and (34), are still true, as is also the last of (32). The first two of the formulae (32), however, are no longer true; just as rot $\nabla\phi$ and div rot $\mathbf{s}$ do not vanish, in general, on a given surface.

The dyadic $\nabla\nabla\phi$ is now not completely symmetric; for its vector, rot $\nabla\phi$, does not vanish identically. But those dyads, whose antecedents and consequents are surface vectors, form a symmetric dyadic; for otherwise the vector of $\nabla\nabla\phi$ would contain a term normal to the surface. Since then rot $\nabla\phi$ is tangential to the surface, the part of $\nabla\nabla\phi$ which is uniplanar and in the tangent plane is a symmetric dyadic.

Applying ∇ to the position vector, $\mathbf{r}$, of the current point on the surface we have

$$\begin{aligned}\nabla\mathbf{r} &= \frac{1}{E}\mathbf{r}_1\mathbf{r}_1 + \frac{1}{G}\mathbf{r}_2\mathbf{r}_2\\ &= \mathbf{a}\,\mathbf{a} + \mathbf{b}\,\mathbf{b} \quad \text{.........................(37)},\end{aligned}$$

where **a**, **b** are the unit tangents to the orthogonal parametric curves. Thus $\nabla\mathbf{r}$ is a unit dyadic for surface vectors.

78. The dyadic $\nabla\mathbf{n}$ for a surface. The dyadic $\nabla\mathbf{n}$ for a given surface will occur frequently in the following pages. If the parametric curves are orthogonal, its value is given by

$$\nabla\mathbf{n} = \frac{1}{E}\mathbf{r}_1\mathbf{n}_1 + \frac{1}{G}\mathbf{r}_2\mathbf{n}_2$$

$$= -\left[\frac{L}{E}\mathbf{a}\,\mathbf{a} + \frac{N}{G}\mathbf{b}\,\mathbf{b} + \frac{M}{H}(\mathbf{a}\,\mathbf{b} + \mathbf{b}\,\mathbf{a})\right] \quad \text{.........(38)},$$

where **a**, **b** as usual denote the unit tangents to the orthogonal parametric curves. The scalar of this dyadic is the negative of the first curvature, or

$$-J = (\nabla\mathbf{n})_s = \nabla\cdot\mathbf{n} = \operatorname{div}\mathbf{n},$$

as we have already seen. The vector of $\nabla\mathbf{n}$ is zero, agreeing with the known result $\operatorname{rot}\mathbf{n} = 0$, and showing that $\nabla\mathbf{n}$ is a *symmetric* dyadic. The second of this dyadic has the value

$$(\nabla\mathbf{n})_2 = \frac{LN - M^2}{H^2}\mathbf{n}\,\mathbf{n} = K\mathbf{n}\,\mathbf{n},$$

and the scalar of this is equal to K. Thus

$$K = (\nabla\mathbf{n})_{2s} \quad \text{......................(39)},$$

or, *the Gaussian curvature is the scalar of the second of* $\nabla\mathbf{n}$.

We may observe that this formula remains true if ∇ is the three-parametric operator for space. For then

$$\nabla\mathbf{n} = \mathbf{i}\frac{\partial\mathbf{n}}{\partial x} + \mathbf{j}\frac{\partial\mathbf{n}}{\partial y} + \mathbf{k}\frac{\partial\mathbf{n}}{\partial z},$$

and therefore $$(\nabla\mathbf{n})_2 = \Sigma\mathbf{i}\left(\frac{\partial\mathbf{n}}{\partial y}\times\frac{\partial\mathbf{n}}{\partial z}\right),$$

and the scalar of this dyadic is equal to K, by (16) of Art. 55.

The function $\overline{\nabla}\phi$ for a given surface, defined in Art. 19 by the equation

$$\overline{\nabla}\phi = -\nabla\phi\cdot\nabla\mathbf{n},$$

is also given by $$\overline{\nabla}\phi = -(\nabla\mathbf{n})\cdot\nabla\phi \quad \text{...................(40)},$$

since $\nabla\mathbf{n}$ is symmetric. It is therefore equal to the direct product of $\nabla\phi$ with $-\nabla\mathbf{n}$, used either as a prefactor or as a postfactor.

Thus $\overline{\nabla}$ may be regarded as the operator obtained by forming the symbolic direct product of $-\nabla\mathbf{n}$ with the operator ∇; or

$$\overline{\nabla} = -(\nabla\mathbf{n})\cdot\nabla.$$

Similarly the function $\nabla^*\phi$, which is given, relative to the lines of curvature as parametric curves, by the equation

$$\nabla^*\phi = \frac{1}{L}\mathbf{r}_1\phi_1 + \frac{1}{N}\mathbf{r}_2\phi_2,$$

may also be expressed

$$\nabla^*\phi = \left(\frac{E}{L}\mathbf{a}\,\mathbf{a} + \frac{G}{N}\mathbf{b}\mathbf{b}\right)\cdot\left(\frac{1}{\sqrt{E}}\mathbf{a}\phi_1 + \frac{1}{\sqrt{G}}\mathbf{b}\phi_2\right)$$
$$= -(\nabla\mathbf{n})^{-1}\cdot\nabla\phi \quad \text{.............................(41)},$$

where $(\nabla\mathbf{n})^{-1}$ is the dyadic which, in direct multiplication with $\nabla\mathbf{n}$, gives $\nabla\mathbf{r}$. It is not the reciprocal of $\nabla\mathbf{n}$ in the sense of Art. 69 but is the *uniplanar reciprocal* of $\nabla\mathbf{n}$, that is to say, its reciprocal relative to the tangent plane, which is the plane of $\nabla\mathbf{n}$.

79. Other geometrical illustrations. The fundamental curvature properties of a surface may be neatly expressed by means of the dyadic $\nabla\mathbf{n}$ for that surface. Take, for instance, the normal curvature κ_n in the direction of the unit surface vector $\mathbf{a}$, and the *torsion* τ *of the geodesic* in this direction. Since $\mathbf{a}$ is the unit tangent and $\mathbf{n}$ the unit principal normal to this geodesic its unit binormal is $\mathbf{t}\times\mathbf{n}$, or $-\mathbf{b}$ with the usual notation. It follows then from the Serret-Frenet formula that

$$\mathbf{a}\cdot\nabla\mathbf{n} = -\kappa_n\mathbf{a} - \tau\mathbf{b},$$

so that

$$\kappa_n = -\mathbf{a}\cdot\nabla\mathbf{n}\cdot\mathbf{a} \quad \text{......................(42)},$$

and

$$\left.\begin{aligned}\tau &= -\mathbf{a}\cdot\nabla\mathbf{n}\cdot\mathbf{b}\\ &= -\mathbf{a}\cdot(\nabla\mathbf{n}\times\mathbf{n})\cdot\mathbf{a}\end{aligned}\right\} \quad \text{..............(43)}.$$

The unit surface vector obtained by a positive rotation θ from the direction of $\mathbf{a}$ is

$$\mathbf{t} = \mathbf{a}\cos\theta + \mathbf{b}\sin\theta,$$

and the unit vector $\mathbf{t}'$ obtained by a further positive rotation of one right angle is

$$\mathbf{t}' = \mathbf{b}\cos\theta - \mathbf{a}\sin\theta.$$

Hence the *normal curvature* in the direction of $\mathbf{t}$ has the value

$$-(\mathbf{a}\cos\theta+\mathbf{b}\sin\theta)\cdot\nabla\mathbf{n}\cdot(\mathbf{a}\cos\theta+\mathbf{b}\sin\theta)$$
$$=\kappa_n\cos^2\theta+\kappa_n'\sin^2\theta+2\tau\sin\theta\cos\theta \quad\ldots(44),$$

where κ_n' is the normal curvature in the direction of $\mathbf{b}$. This formula was found by a different method in Art. 27. The *torsion of the geodesic* in the direction of $\mathbf{t}$ has the value

$$-\mathbf{t}\cdot\nabla\mathbf{n}\cdot\mathbf{t}'=-(\mathbf{a}\cos\theta+\mathbf{b}\sin\theta)\cdot\nabla\mathbf{n}\cdot(\mathbf{b}\cos\theta-\mathbf{a}\sin\theta)$$
$$=(\cos^2\theta-\sin^2\theta)\,\tau+\sin\theta\cos\theta\,(\kappa_n'-\kappa_n),$$

as found in Art. 26.

Again, *conjugate directions* on the surface are such that the derivative of $\mathbf{n}$ in either direction is perpendicular to the other. Hence the directions of the surface vectors $\mathbf{c}$ and $\mathbf{d}$ will be conjugate provided

$$\mathbf{c}\cdot\nabla\mathbf{n}\cdot\mathbf{d}=0.$$

Asymptotic directions are self-conjugate. Hence, if the direction of $\mathbf{d}$ is asymptotic,

$$\mathbf{d}\cdot\nabla\mathbf{n}\cdot\mathbf{d}=0,$$

in agreement with the property that the normal curvature vanishes for an asymptotic direction. The differential equation of the asymptotic lines is thus

$$d\mathbf{r}\cdot\nabla\mathbf{n}\cdot d\mathbf{r}=0.$$

A *principal direction* $\mathbf{e}$ on the surface is such that the derivative of $\mathbf{n}$ in that direction is parallel to $\mathbf{e}$ and therefore perpendicular to $\mathbf{n}\times\mathbf{e}$, so that

$$\mathbf{e}\cdot(\nabla\mathbf{n}\times\mathbf{n})\cdot\mathbf{e}=0.$$

This agrees with the property that the torsion of the geodesic tangent to a line of curvature is zero. The differential equation of the lines of curvature may be expressed

$$d\mathbf{r}\cdot(\nabla\mathbf{n}\times\mathbf{n})\cdot d\mathbf{r}=0.$$

EXAMPLES VII

1. Show that
$$(\mathbf{a}\times\mathbf{b})\times\mathbf{r}=(\mathbf{b\,a}-\mathbf{a\,b})\cdot\mathbf{r},$$
$$\mathbf{r}\times(\mathbf{a}\times\mathbf{b})=\mathbf{r}\cdot(\mathbf{b\,a}-\mathbf{a\,b}),$$
$$(\mathbf{a}\times\mathbf{b})\times\mathrm{I}=\mathbf{b\,a}-\mathbf{a\,b}.$$

2. Prove that
$$(\mathrm{I}\times\mathbf{a})\cdot\Phi=\mathbf{a}\times\Phi,$$
and
$$(\mathbf{a}\times\mathrm{I})\cdot\Phi=\mathbf{a}\times\Phi.$$

3. If $\Phi^n=\Phi\cdot\Phi\cdot\Phi\cdot\ldots$ to n factors, show that
$$(\Phi^n)^{-1}=(\Phi^{-1})^n.$$
Also prove that
$$(\Phi^{-1})_c=(\Phi_c)^{-1},$$
and
$$(\Phi^n)_c=(\Phi_c)^n.$$

4. Prove the formula
$$\Phi\cdot\mathbf{r}=\tfrac{1}{2}(\Phi+\Phi_c)\cdot\mathbf{r}-\tfrac{1}{2}\Phi_v\times\mathbf{r}.$$

5. Show that $\Phi\cdot\Phi_c$ is symmetric, and that
$$(\Phi\times\mathbf{u})_c=-\mathbf{u}\times\Phi_c.$$

6. Prove that $(\Phi\cdot\Psi\cdot\Omega)_s=(\Psi\cdot\Omega\cdot\Phi)_s=(\Omega\cdot\Phi\cdot\Psi)_s.$

7. Show that
$$\mathbf{l}(\mathbf{m}\times\mathbf{n})+\mathbf{m}(\mathbf{n}\times\mathbf{l})+\mathbf{n}(\mathbf{l}\times\mathbf{m})=[\mathbf{l},\mathbf{m},\mathbf{n}]\,\mathrm{I}$$
$$=(\mathbf{m}\times\mathbf{n})\,\mathbf{l}+(\mathbf{n}\times\mathbf{l})\,\mathbf{m}+(\mathbf{l}\times\mathbf{m})\,\mathbf{n}.$$

8. Prove that $(\Phi^{-1})_2=(\Phi_2)^{-1}$ and $(\Phi^{-1})_3=(\Phi_3)^{-1}.$

9. Show that
$$(\Phi\cdot\Psi)_2=\Phi_2\cdot\Psi_2,$$
and
$$(\Phi\cdot\Psi)_3=\Phi_3\Psi_3.$$
Also that
$$(\Phi^n)_2=(\Phi_2)^n \text{ and } (\Phi^n)_3=(\Phi_3)^n.$$

10. If $\Phi\cdot\mathbf{r}=0$ for three non-coplanar values of $\mathbf{r}$, show that $\Phi=0$.

11. A necessary and sufficient condition that an anti-symmetric dyadic be zero is that the vector of the dyadic be zero.

12. Show that
$$\Phi_2 \overset{\times}{\times} \Phi_2=\Phi_3{}^2\Phi,$$
and
$$(\Phi+\Psi)_2=\Phi_2+\Phi\overset{\times}{\times}\Psi+\Psi_2.$$

13. Show that $(\Phi+\mathbf{a\,b})_3=\Phi_3+\mathbf{a}\cdot\Phi_2\cdot\mathbf{b}.$

14. Prove that $\nabla\mathbf{s}$ and $\mathbf{s}\nabla$ are invariant with respect to the choice of coordinates.

15. Verify the formulae (32), and show that
$$\nabla\times(\nabla\times\Phi)=\nabla(\nabla\cdot\Phi)-\nabla^2\Phi.$$

16. Verify the formulae (33).

17. Show that, if ∇ is the operator for space,

$$\nabla\cdot(\phi\mathrm{I})=\nabla\phi,$$
$$\nabla\times(\phi\mathrm{I})=\nabla\phi\times\mathrm{I},$$
$$\nabla\cdot(\mathrm{I}\times\mathbf{s})=\nabla\times\mathbf{s}=\mathrm{rot}\,\mathbf{s},$$
$$\nabla\times(\mathrm{I}\times\mathbf{s})=\mathbf{s}\nabla-\mathrm{I}\,\mathrm{div}\,\mathbf{s}.$$

18. Prove the formulae

$$\nabla\times(\mathbf{u}\times\mathbf{v})=\nabla\cdot(\mathbf{v}\,\mathbf{u}-\mathbf{u}\,\mathbf{v}),$$
$$\nabla\cdot(\mathbf{u}\,\mathbf{v})=\nabla\cdot\mathbf{u}\,\mathbf{v}+\mathbf{u}\cdot\nabla\mathbf{v},$$
$$\nabla\times(\mathbf{u}\,\mathbf{v})=\nabla\times\mathbf{u}\,\mathbf{v}-\mathbf{u}\times\nabla\mathbf{v}.$$

19. Show that $\quad \nabla\mathbf{s}=\frac{1}{2}(\nabla\mathbf{s}+\mathbf{s}\nabla)-\frac{1}{2}\mathrm{I}\times(\nabla\times\mathbf{s}).$

20. Prove the formulae

$$\nabla\cdot(\Phi\cdot\mathbf{u})=(\nabla\cdot\Phi)\cdot\mathbf{u}+\Phi:\nabla\mathbf{u},$$
$$\nabla\times(\Phi\times\mathbf{u})=(\nabla\times\Phi)\times\mathbf{u}-\nabla\mathbf{u}\overset{\times}{\times}\Phi.$$

21. Show that $\quad \mathrm{I}_2=\mathrm{I},\quad \mathrm{I}_3=1,$

and $\quad \Phi:\mathrm{I}=\mathrm{I}:\Phi=\Phi_s.$

22*. By the substitution $\phi=\mathbf{u}\cdot\mathbf{d}$ in (43) of Art. 49, $\mathbf{d}$ being a constant vector, deduce the theorem

$$\iint\mathbf{n}\times\nabla\mathbf{u}\,dS=\int_{\circ}d\mathbf{r}\,\mathbf{u}.$$

23. Apply Stokes's theorem to the vector $\Phi\cdot\mathbf{d}$, where $\mathbf{d}$ is constant, and deduce the formula

$$\iint\mathbf{n}\cdot\nabla\times\Phi\,dS=\int_{\circ}d\mathbf{r}\cdot\Phi=\int_{\circ}\Phi_c\cdot d\mathbf{r}.$$

24. Applying Stokes's theorem to the vector $\mathbf{u}\times\mathbf{d}$, where $\mathbf{d}$ is constant, deduce the formula

$$\int_{\circ}d\mathbf{r}\times\mathbf{u}=\iint[\mathbf{n}\cdot(\mathbf{u}\nabla)-\mathbf{n}\nabla\cdot\mathbf{u}]\,dS.$$

25. In (47) of Art. 51, put $\phi=\mathbf{u}\cdot\mathbf{d}$, where $\mathbf{d}$ is constant, and deduce the formula

$$\iiint\nabla\mathbf{u}\,dV=\iint\mathbf{n}\,\mathbf{u}\,dS.$$

26. Applying the Divergence Theorem to the vector $\Phi\cdot\mathbf{d}$, where $\mathbf{d}$ is constant, deduce the theorem

$$\iiint\nabla\cdot\Phi\,dV=\iint\mathbf{n}\cdot\Phi\,dS.$$

27. In (48) of Art. 51, put $\mathbf{F}=\Phi\cdot\mathbf{d}$, where $\mathbf{d}$ is constant, and deduce the formula

$$\iiint\nabla\times\Phi\,dV=\iint\mathbf{n}\times\Phi\,dS.$$

* The theorems of Exx. 22–27 are proved otherwise in the author's *Advanced Vector Analysis*, Arts. 86–87.

28. If Φ is small, that is to say, one vector in each dyad is small, prove that, neglecting small quantities of higher order, $\mathrm{I}+\Phi$ and $\mathrm{I}-\Phi$ are reciprocal, and that

$$(\mathrm{I}+\Phi)^n=\mathrm{I}+n\Phi.$$

29. If $\mathbf{s}$ is small, show that $\mathrm{I}+\nabla\mathbf{s}$ and $\mathrm{I}-\nabla\mathbf{s}$ are reciprocal.

30. Order of directional differentiations. A proof of the theorem of Art. 5 may be given, independently of the use of coordinates, and at the same time the theorem may be extended to any two directions.

Let $\mathbf{a}$, $\mathbf{b}$ be the unit tangents to an orthogonal system of curves. Then, in virtue of (33),

$$\nabla(\mathbf{a}\cdot\nabla\phi)=\nabla\mathbf{a}\cdot\nabla\phi+\nabla\nabla\phi\cdot\mathbf{a},$$

and therefore $\quad \mathbf{b}\cdot\nabla(\mathbf{a}\cdot\nabla\phi)=\mathbf{b}\cdot\nabla\mathbf{a}\cdot\nabla\phi+\mathbf{b}\cdot\nabla\nabla\phi\cdot\mathbf{a}.$

Similarly $\quad \mathbf{a}\cdot\nabla(\mathbf{b}\cdot\nabla\phi)=\mathbf{a}\cdot\nabla\mathbf{b}\cdot\nabla\phi+\mathbf{a}\cdot\nabla\nabla\phi\cdot\mathbf{b}.$

Consequently, since the part of $\nabla\nabla\phi$ which consists only of surface vectors is symmetric (Art. 77), we have on subtraction

$$\begin{aligned}\mathbf{b}\cdot\nabla(\mathbf{a}\cdot\nabla\phi)-\mathbf{a}\cdot\nabla(\mathbf{b}\cdot\nabla\phi)&=(\mathbf{b}\cdot\nabla\mathbf{a}-\mathbf{a}\cdot\nabla\mathbf{b})\cdot\nabla\phi\\&=(\mathbf{b}\operatorname{div}\mathbf{a}-\mathbf{a}\operatorname{div}\mathbf{b})\cdot\nabla\phi\\&=(\gamma\mathbf{a}+\gamma'\mathbf{b})\cdot\nabla\phi,\end{aligned}$$

which agrees with (26) of Art. 5.

Suppose that $\mathbf{c}$ and $\mathbf{d}$ are the unit tangents to any two families of curves, cutting at a variable angle ω. Then, by the same argument as above,

$$\mathbf{d}\cdot\nabla(\mathbf{c}\cdot\nabla\phi)-\mathbf{c}\cdot\nabla(\mathbf{d}\cdot\nabla\phi)=(\mathbf{d}\cdot\nabla\mathbf{c}-\mathbf{c}\cdot\nabla\mathbf{d})\cdot\nabla\phi.$$

If, now, ω is the angle of rotation from $\mathbf{c}$ to $\mathbf{d}$ in the positive sense,

$$\mathbf{c}\times\mathbf{d}=(\sin\omega)\,\mathbf{n}.$$

Consequently, on taking the rotation of both members,

$$\mathbf{d}\cdot\nabla\mathbf{c}-\mathbf{c}\cdot\nabla\mathbf{d}+\mathbf{c}\operatorname{div}\mathbf{d}-\mathbf{d}\operatorname{div}\mathbf{c}=(\cos\omega)\,\nabla\omega\times\mathbf{n}.$$

The above difference of the two second derivatives may therefore be expressed

$$(\mathbf{d}\operatorname{div}\mathbf{c}-\mathbf{c}\operatorname{div}\mathbf{d}+\cos\omega\,\nabla\omega\times\mathbf{n})\cdot\nabla\phi.$$

31. Using orthogonal parametric curves, show that

$$\nabla\times\nabla\mathbf{r}=\frac{N}{G}\,\mathbf{a}\,\mathbf{b}-\frac{L}{E}\mathbf{b}\,\mathbf{a}+\frac{M}{H}(\mathbf{a}\,\mathbf{a}-\mathbf{b}\,\mathbf{b}),$$

and hence that

$$(\nabla\times\nabla\mathbf{r})^2=-\frac{(LN-M^2)}{H^2}(\mathbf{a}\,\mathbf{a}+\mathbf{b}\,\mathbf{b})=-K\nabla\mathbf{r}.$$

Consequently $\quad K=-(\nabla\times\nabla\mathbf{r})^2/\nabla\mathbf{r}.$

Also show that $\quad (\nabla\times\nabla\mathbf{r})_2=\dfrac{LN-M^2}{H^2}\mathbf{n}\,\mathbf{n}=K\mathbf{n}\,\mathbf{n},$

and hence that $\quad K=(\nabla\times\nabla\mathbf{r})_{2s}.$

32. Show that, for any choice of parameters u, v on a given surface, the operator ∇ for that surface gives†

$$\nabla \mathbf{n} = -\frac{1}{H^2}[\mathbf{r}_1(G\mathbf{n}_1 - F\mathbf{n}_2) + \mathbf{r}_2(E\mathbf{n}_2 - F\mathbf{n}_1)],$$

and therefore, $\nabla \mathbf{n}$ being symmetric,

$$\begin{aligned}\bar{\nabla} = -\nabla \mathbf{n} \cdot \nabla &= -\frac{1}{H^2}\left[\mathbf{n}_1\left(G\frac{\partial}{\partial u} - F\frac{\partial}{\partial v}\right) + \mathbf{n}_2\left(E\frac{\partial}{\partial v} - F\frac{\partial}{\partial u}\right)\right] \\ &= \frac{1}{H^2}\mathbf{r}_1\left[(GJ-N)\frac{\partial}{\partial u} - (FJ-M)\frac{\partial}{\partial v}\right] \\ &\qquad + \frac{1}{H^2}\mathbf{r}_2\left[(EJ-L)\frac{\partial}{\partial v} - (FJ-M)\frac{\partial}{\partial u}\right].\end{aligned}$$

Similarly the uniplanar reciprocal of $\nabla \mathbf{n}$ is given by

$$(\nabla \mathbf{n})^{-1} = \frac{-1}{LN-M^2}[\mathbf{r}_1(N\mathbf{r}_1 - M\mathbf{r}_2) + \mathbf{r}_2(L\mathbf{r}_2 - M\mathbf{r}_1)],$$

and

$$\nabla^* = -(\nabla \mathbf{n})^{-1} \cdot \nabla = \frac{1}{LN-M^2}\left[\mathbf{r}_1\left(N\frac{\partial}{\partial u} - M\frac{\partial}{\partial v}\right) + \mathbf{r}_2\left(L\frac{\partial}{\partial v} - M\frac{\partial}{\partial u}\right)\right],$$

which is obtainable from ∇, replacing the first order magnitudes E, F, G by the second order magnitudes L, M, N respectively. Verify the identity

$$K\nabla^* = J\nabla - \bar{\nabla}.$$

33. Show that functions such as $\bar{\nabla} \cdot (\phi \mathbf{u})$, $\bar{\nabla} \times (\phi \mathbf{u})$, and $\nabla^* \cdot (\mathbf{u} \times \mathbf{v})$, formed with the above operators, obey similar formulae of expansion to those of Art. 3 for the operator ∇.

34. Prove that, with the usual notation for any surface,

$$\bar{\nabla} \cdot \nabla \mathbf{r} = (J^2 - 2K)\mathbf{n}, \quad \nabla^* \cdot \nabla \mathbf{r} = 2\mathbf{n}.$$

35. Show that, on a given surface,

$$\nabla \mathbf{n} : \nabla \mathbf{s} = (\nabla \mathbf{n} \cdot \nabla \mathbf{s})_s = -\bar{\nabla} \cdot \mathbf{s}.$$

36. Prove that, for space differential invariants,

$$\mathbf{s} \times \operatorname{rot} \mathbf{n} = \nabla \mathbf{n} \cdot \mathbf{s} - \mathbf{s} \cdot \nabla \mathbf{n}.$$

† *Quarterly Journal*, Vol. 50, pp. 277–278.

CHAPTER VIII

FAMILIES OF CURVES AND FUNCTIONS OF DIRECTION ON A SURFACE

CENTRAL QUADRICS

80. Central quadric surfaces. Before proceeding to discuss further properties of a family of curves on a given surface, we may draw attention to a convenient representation of some of the fundamental properties of central quadric surfaces and central conics, in terms of dyadics associated with them.

If $\mathbf{r}$ is the position vector of the current point P relative to a fixed origin O, and Φ is a complete dyadic independent of $\mathbf{r}$, the equation

$$\mathbf{r} \cdot \Phi \cdot \mathbf{r} = \text{const.}$$

represents a quadric surface with centre at the origin. It is clearly of the second degree in $\mathbf{r}$, and therefore in the coordinates of P. And, if a point $\mathbf{s}$ lies on the surface, so does the point $-\mathbf{s}$, showing that O is the centre of the quadric. The second member of the above equation may be made unity; for the dyadic may be divided by the value of the constant. Further, we may take Φ to be symmetric. For any dyadic is expressible as the sum of a symmetric part Ψ and an anti-symmetric part Ω (Art. 70); and the latter contributes nothing to the value of $\mathbf{r} \cdot \Phi \cdot \mathbf{r}$. For

$$\mathbf{r} \cdot \Omega = -\Omega \cdot \mathbf{r},$$

and consequently $$\mathbf{r} \cdot \Omega \cdot \mathbf{r} = -\mathbf{r} \cdot \Omega \cdot \mathbf{r},$$

so that the value of this expression is zero. Thus the equation of the central quadric may be put in the form

$$\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1 \qquad \text{...(1)},$$

where Φ is a symmetric dyadic.

Consider any infinitesimal displacement on the quadric surface from the point $\mathbf{r}$ to the point $\mathbf{r} + d\mathbf{r}$. Then, since Φ is constant, it follows from (1) that

$$d\mathbf{r} \cdot \Phi \cdot \mathbf{r} + \mathbf{r} \cdot \Phi \cdot d\mathbf{r} = 0,$$

and therefore, since Φ is symmetric,

$$d\mathbf{r}\cdot\Phi\cdot\mathbf{r}=0.$$

Thus any infinitesimal displacement $d\mathbf{r}$ on the surface, from the point $\mathbf{r}$, is perpendicular to $\Phi\cdot\mathbf{r}$. It follows that $\Phi\cdot\mathbf{r}$ is parallel to the *normal* at $\mathbf{r}$, and the equation of the *tangent plane* at that point is

$$(\mathbf{R}-\mathbf{r})\cdot\Phi\cdot\mathbf{r}=0,$$

$\mathbf{R}$ being the current point on this plane. Since the point $\mathbf{r}$ lies on the surface, this equation may be more briefly expressed

$$\mathbf{R}\cdot\Phi\cdot\mathbf{r}=1 \quad\text{.......................}(2).$$

Hence, in the usual manner, it may be shown that the *polar plane* of a point $\mathbf{d}$ is given by the equation

$$\mathbf{R}\cdot\Phi\cdot\mathbf{d}=1 \quad\text{.......................}(3).$$

The *diametral plane*, for chords parallel to the vector $\mathbf{h}$, is the plane through the centre parallel to the tangent plane at the end of the diameter whose direction is $\mathbf{h}$. It is thus perpendicular to $\Phi\cdot\mathbf{h}$, and its equation is

$$\mathbf{R}\cdot\Phi\cdot\mathbf{h}=0 \quad\text{.......................}(4).$$

It is important to notice that the sum of the squares of the reciprocals of three mutually perpendicular radii of the quadric is equal to the scalar of Φ. For, if $\mathbf{a}$ is a unit vector, it follows from (1) that $\mathbf{a}\cdot\Phi\cdot\mathbf{a}$ is equal to the inverse square of the radius in the direction of $\mathbf{a}$. But $\Sigma\mathbf{a}\cdot\Phi\cdot\mathbf{a}$ for three mutually perpendicular directions is equal to Φ_s (Art. 68); and the truth of the above statement is then apparent.

It may also be shown that the *reciprocal* of the quadric

$$\mathbf{r}\cdot\Phi\cdot\mathbf{r}=c,$$

relative to a concentric sphere of radius a, is the quadric

$$\mathbf{r}\cdot\Phi^{-1}\cdot\mathbf{r}=c',$$

where Φ^{-1} is the reciprocal dyadic to Φ, and*

$$cc'=a^4.$$

81. Central conics. Next suppose that all the points, $\mathbf{r}$, considered lie in a plane through the origin perpendicular to the vector $\mathbf{n}$, so that $\mathbf{r}\cdot\mathbf{n}=0$. Then the equation

$$\mathbf{r}\cdot\Phi\cdot\mathbf{r}=1 \quad\text{.........................}(5)$$

* For a fuller account of quadric surfaces see *Advanced Vector Analysis*, pp. 93–100.

represents a conic whose centre is the origin. As before the dyadic Φ may be assumed symmetric. It may also be taken as consisting only of dyads whose antecedents and consequents are parallel to the plane of the conic. For any antecedents or consequents normal to this plane contribute nothing to the value of $\mathbf{r} \cdot \Phi \cdot \mathbf{r}$, in virtue of the relation $\mathbf{r} \cdot \mathbf{n} = 0$. It then follows as above that $\Phi \cdot \mathbf{r}$ is parallel to the normal at the point $\mathbf{r}$, and that the tangent at this point is given by (2). Similarly the polar of the point $\mathbf{d}$ is given by (3). Also $\mathbf{c}$ and $\mathbf{d}$ will be parallel to conjugate diameters provided

$$\mathbf{c} \cdot \Phi \cdot \mathbf{d} = 0 \qquad\qquad (6).$$

By the same argument as for a quadric surface it follows that the sum of the inverse squares of any two perpendicular radii of the conic is equal to Φ_s, the dyadic Φ consisting only of vectors parallel to the plane of the conic.

Illustrations of the above occur in the elementary theory of curvature of a surface. We have seen that the normal curvature at a point P, in the direction of the unit vector $\mathbf{d}$, has the value $-\mathbf{d} \cdot \nabla \mathbf{n} \cdot \mathbf{d}$. It is therefore equal to the inverse square of the radius of the conic

$$\mathbf{r} \cdot \nabla \mathbf{n} \cdot \mathbf{r} = -1 \qquad\qquad (7),$$

in that direction, the centre of the conic being P, and its plane the tangent plane at P. This conic is of course Dupin's *indicatrix*. The normal curvature vanishes when the radius of this conic becomes infinite, which is the case for the directions of the asymptotes of the conic. The sum of the normal curvatures in any two perpendicular directions at P, being equal to the sum of the inverse squares of the radii in these directions, has the value of the scalar of $-\nabla \mathbf{n}$, which is $-\operatorname{div} \mathbf{n}$, as already proved.

Again, it was shown that the torsion of the geodesic, in the direction of the unit vector $\mathbf{d}$ at the point P, has the value $-\mathbf{d} \cdot (\nabla \mathbf{n} \times \mathbf{n}) \cdot \mathbf{d}$. It is therefore equal to the inverse square of the radius of the conic

$$\mathbf{r} \cdot (\nabla \mathbf{n} \times \mathbf{n}) \cdot \mathbf{r} = -1 \qquad\qquad (8),$$

in that direction. The centre of this conic is the point P, and its plane is the tangent plane at that point. The torsion of the geodesic vanishes for the directions of the asymptotes of this conic, which are therefore the principal directions at P. Thus (8) repre-

sents a rectangular hyperbola; and, since the sum of the inverse squares of the radii thus vanishes for two perpendicular directions, it follows that the sum of the torsions vanishes for any two geodesics which cut orthogonally at P. In agreement with this result, the scalar of $-\nabla\mathbf{n}\times\mathbf{n}$ vanishes. For, if the parametric curves are the lines of curvature, with unit tangents $\mathbf{a}$ and $\mathbf{b}$,

$$-\nabla\mathbf{n}\times\mathbf{n}=\left(\frac{L}{E}\mathbf{a}\,\mathbf{a}+\frac{N}{G}\mathbf{b}\,\mathbf{b}\right)\times\mathbf{n}=-\frac{L}{E}\mathbf{a}\,\mathbf{b}+\frac{N}{G}\mathbf{b}\,\mathbf{a},$$

and the scalar of this is obviously equal to zero.

Family of Curves on a Surface*

82. Tendency in any direction. First conic. Just as the curvature properties of a surface at any point are associated with the above conics, so the fundamental properties of a family of curves on the surface are, at each point, associated with three central conics determined by the family of curves. The differential invariants employed in this connection are, of course, the two-parametric invariants for the given surface.

Let $\mathbf{t}$ be the unit tangent to the curve at any point P. This vector is a point-function for the surface. Its derivative in the direction of the unit surface vector $\mathbf{a}$ is given by $\mathbf{a}\cdot\nabla\mathbf{t}$. The resolved part of this derivative in the direction of $\mathbf{a}$ we shall call the *tendency* of the family in that direction. It has the value $\mathbf{a}\cdot\nabla\mathbf{t}\cdot\mathbf{a}$. From the definition of div $\mathbf{t}$ in terms of orthogonal coordinates it then follows that:

The sum of the tendencies for any two perpendicular directions at a point is invariant, and equal to the divergence of the family of curves at that point.

If now we introduce the conic

$$\mathbf{r}\cdot\nabla\mathbf{t}\cdot\mathbf{r}=1\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

whose centre is at the point P, and whose plane is the tangent plane at that point, it is clear that *the tendency of the family in any direction is equal to the inverse square of the radius of the conic* (9) *in that direction, having the value zero for the directions of the*

* The substance of Arts. 82–84 was given by the author in a paper on "Some Properties of a Family of Curves on a Surface," *Proc. Edin. Math. Soc.*, Vol. 1 (1928), pp. 160–165.

asymptotes. For the direction inclined at an angle ϕ to a principal axis of the conic, the tendency T is given by

$$T = T_1 \cos^2 \phi + T_2 \sin^2 \phi \quad\quad (10),$$

where T_1 is the tendency in the direction of the above axis, and T_2 that for the perpendicular direction. The sum of the tendencies in two perpendicular directions has the value $T_1 + T_2$, which is therefore equal to div $\mathbf{t}$. The value of this invariant also follows from the fact that the sum of the inverse squares of two perpendicular radii of the conic (9) is equal to the scalar of $\nabla \mathbf{t}$, which is div $\mathbf{t}$.

It is clear that one of the asymptotes of the conic (9) is the tangent to the curve at the point P. For, since $\mathbf{t}$ is a unit vector, it is perpendicular to its derivative $\mathbf{t} \cdot \nabla \mathbf{t}$; and consequently the tendency in the direction of $\mathbf{t}$ is zero. The tendency in the direction perpendicular to $\mathbf{t}$ is therefore div $\mathbf{t}$. This may be called the *principal tendency.* The two directions of zero tendency will be at right angles provided div $\mathbf{t}$ is zero. Thus:

The locus of points at which the two directions of zero tendency are perpendicular is the line of striction of the family of curves.

The direction of the other asymptote of the conic (9), and an expression for the tendency in any direction in terms of its inclination to that of $\mathbf{t}$, may be found as follows. Consider the direction inclined at an angle θ to that of $\mathbf{t}$, obtained by a positive rotation θ from $\mathbf{t}$ about the normal. The unit vector in this direction is $\mathbf{t} \cos \theta + \mathbf{t}' \sin \theta$, where $\mathbf{t}'$ is the unit surface vector $\mathbf{n} \times \mathbf{t}$, which is the unit tangent to the orthogonal trajectory of the curves. The tendency T in the direction θ is then given by

$$T = (\mathbf{t} \cos \theta + \mathbf{t}' \sin \theta) \cdot \nabla \mathbf{t} \cdot (\mathbf{t} \cos \theta + \mathbf{t}' \sin \theta).$$

Now, since $\mathbf{t}$ is perpendicular to its derivatives, the first term in the last factor contributes nothing to the value of the product. Thus

$$\begin{aligned} T &= \sin \theta \left[\sin \theta \left(\mathbf{t}' \cdot \nabla \mathbf{t} \cdot \mathbf{t}' \right) + \cos \theta \left(\mathbf{t} \cdot \nabla \mathbf{t} \cdot \mathbf{t}' \right) \right] \\ &= \sin \theta \left(\sin \theta \operatorname{div} \mathbf{t} - \cos \theta \operatorname{div} \mathbf{t}' \right) \quad\quad (11), \end{aligned}$$

since $\mathbf{t} \cdot \nabla \mathbf{t} \cdot \mathbf{t}'$ is the geodesic curvature of the family $\mathbf{t}$. Consequently the directions of the asymptotes are given by

$$\sin \theta = 0 \quad \text{and} \quad \tan \theta = \frac{\operatorname{div} \mathbf{t}'}{\operatorname{div} \mathbf{t}} \quad\quad (12).$$

The second asymptote is therefore in the direction of the vector $\mathbf{t}\operatorname{div}\mathbf{t}+\mathbf{t}'\operatorname{div}\mathbf{t}'$, which is the tangential component of the vector curvature of the orthogonal system of which the given family forms part. The tangent of the angle between the two directions of zero tendency is the ratio of the divergence of the family of orthogonal trajectories to that of the given family.

It follows from (12) that the asymptotes are perpendicular where $\operatorname{div}\mathbf{t}$ vanishes, and that they are coincident where $\operatorname{div}\mathbf{t}'$ is zero. But $-\operatorname{div}\mathbf{t}'$ is the geodesic curvature of the given family at the point P. Hence:

The locus of points at which the two directions of zero tendency are coincident is the line of normal curvature of the family of curves.

The asymptotes are coincident at all points for a family of geodesics; and they are perpendicular at all points for a family of parallels.

83. Moment of the family for any direction. Second conic. The moment of a family of curves, as defined in Art. 10, is the moment for the direction of the orthogonal trajectory. Similarly we may define the moment of the family for any surface direction

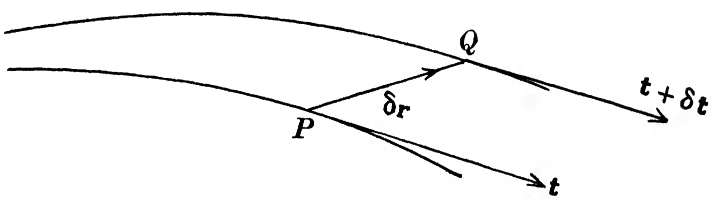

Fig. 6.

at the point. Let $\mathbf{t}$ be the unit tangent to the curve at P, and $\mathbf{t}+\delta\mathbf{t}$ that to the curve through an adjacent point Q in any direction from P, the elementary vector PQ being $\delta\mathbf{r}$. Then

$$\delta\mathbf{t}=\delta\mathbf{r}\cdot\nabla\mathbf{t},$$

neglecting terms of higher order than the first. The mutual moment of the tangents at P and Q is the resolved part, in the direction of $\mathbf{t}$, of the moment of $\mathbf{t}+\delta\mathbf{t}$ about P. Consequently it has the value

$$\delta\mathbf{r}\times(\mathbf{t}+\delta\mathbf{t})\cdot\mathbf{t}=\delta\mathbf{r}\times\delta\mathbf{t}\cdot\mathbf{t}=\delta\mathbf{r}\times(\delta\mathbf{r}\cdot\nabla\mathbf{t})\cdot\mathbf{t}=\delta\mathbf{r}\cdot(\nabla\mathbf{t}\times\mathbf{t})\cdot\delta\mathbf{r},$$

neglecting terms of higher order than the second. Now let Q tend to coincidence with P; and let $\mathbf{a}$ be the unit vector in the limiting direction of PQ. Then the limiting value of the quotient of the above moment by $(\delta\mathbf{r})^2$ is the function

$$M = \mathbf{a} \cdot (\nabla\mathbf{t} \times \mathbf{t}) \cdot \mathbf{a} \quad \text{.....................(13)}.$$

We shall call this the moment of the family of curves at the point P for the direction of $\mathbf{a}$. If then we introduce the conic

$$\mathbf{r} \cdot (\nabla\mathbf{t} \times \mathbf{t}) \cdot \mathbf{r} = 1 \quad \text{.....................(14)},$$

whose centre is at the point P, and whose plane is the tangent plane at that point, it is clear from (13) that *the moment of the family for any direction at P is equal to the inverse square of the radius of the conic* (14) *in that direction, having the value zero for the directions of the asymptotes.*

For a direction inclined at an angle ϕ to a principal axis of the conic, the moment of the family is given by

$$M = M_1 \cos^2\phi + M_2 \sin^2\phi,$$

where M_1 is the moment for the direction of the above axis, and M_2 that for the perpendicular direction. The sum of the moments for two perpendicular directions has the value $M_1 + M_2$, and is therefore invariant. Being the sum of the inverse squares of two perpendicular radii of (14), it is equal to the scalar of $\nabla\mathbf{t} \times \mathbf{t}$. But, in terms of orthogonal coordinates,

$$(\nabla\mathbf{t} \times \mathbf{t})_s = \left(\Sigma \frac{1}{E}\mathbf{r}_1 \ \mathbf{t}_1 \times \mathbf{t}\right)_s = \Sigma \frac{1}{E}\mathbf{r}_1 \times \mathbf{t}_1 \cdot \mathbf{t} = \mathbf{t} \cdot \operatorname{rot} \mathbf{t} \quad \text{...(15)}.$$

This quantity may be called the *total moment* (or briefly the *moment*) of the family at that point. Thus:

The sum of the moments for two perpendicular directions at a point is invariant, having the value $\mathbf{t} \cdot \operatorname{rot} \mathbf{t}$.

Now we have already seen that $\mathbf{t} \cdot \operatorname{rot} \mathbf{t}$ is the value of the moment for the direction perpendicular to $\mathbf{t}$ (Art. 10). Consequently the direction of $\mathbf{t}$ is one of zero moment. This is also clear from the fact that the shortest distance between the tangents to a curve at near-by points is a small quantity of higher order than the second. The tangent to the curve at P is therefore one of the asymptotes of the conic (14). If the two directions of zero moment

are at right angles, the total moment is zero; and conversely. Thus:

The locus of points at which the two directions of zero moment are perpendicular is the line of zero moment of the family of curves.

The direction of the second asymptote of (14), and an expression for the moment in any direction in terms of its inclination θ to that of $\mathbf{t}$, may be found as follows. With the same notation as in the preceding article, the moment in the direction inclined at an angle θ to that of $\mathbf{t}$ is given by

$$M = (\mathbf{t}\cos\theta + \mathbf{t}'\sin\theta)\cdot(\nabla\mathbf{t}\times\mathbf{t})\cdot(\mathbf{t}\cos\theta + \mathbf{t}'\sin\theta).$$

Since the scalar triple products involving the repeated factor $\mathbf{t}$ are all zero, the first term in the last factor contributes nothing to the value of the above expression. Consequently

$$M = \sin\theta\,[\sin\theta\ \ \mathbf{t}'\cdot(\nabla\mathbf{t}\times\mathbf{t})\cdot\mathbf{t}' + \cos\theta\ \ \mathbf{t}\cdot(\nabla\mathbf{t}\times\mathbf{t})\cdot\mathbf{t}'].$$

Now $\mathbf{t}'\cdot(\nabla\mathbf{t}\times\mathbf{t})\cdot\mathbf{t}'$ is the moment for the direction perpendicular to $\mathbf{t}$, and therefore has the value $\mathbf{t}\cdot\operatorname{rot}\mathbf{t}$. Also

$$\mathbf{t}\cdot(\nabla\mathbf{t}\times\mathbf{t})\cdot\mathbf{t}' = \mathbf{t}\cdot\nabla\mathbf{t}\cdot\mathbf{n},$$

and is therefore equal to the normal curvature of the surface in the direction of $\mathbf{t}$, whose value may also be expressed $-\mathbf{t}'\cdot\operatorname{rot}\mathbf{t}$. Thus

$$M = \sin\theta\,(\sin\theta\ \ \mathbf{t}\cdot\operatorname{rot}\mathbf{t} - \cos\theta\ \ \mathbf{t}'\cdot\operatorname{rot}\mathbf{t}) \quad\text{......(16).}$$

Consequently the directions of zero moment are given by

$$\sin\theta = 0 \quad\text{and}\quad \tan\theta = \frac{\mathbf{t}'\cdot\operatorname{rot}\mathbf{t}}{\mathbf{t}\cdot\operatorname{rot}\mathbf{t}} \quad\text{............(17).}$$

The second of these is the direction *conjugate* to that of $\mathbf{t}$; for it is the direction of the tangential component of rot $\mathbf{t}$ (Art. 9). From (17) it follows again that the two directions of zero moment are perpendicular where $\mathbf{t}\cdot\operatorname{rot}\mathbf{t}$ is zero, and also that the two directions are coincident where $\mathbf{t}'\cdot\operatorname{rot}\mathbf{t}$ vanishes. Thus:

The locus of the points at which the two directions of zero moment are coincident is the line of tangential curvature of the family of curves.

At such points the direction of the curve is an asymptotic direction on the surface.

84. Swerve of the family. Third conic. Consider again the derivative of $\mathbf{t}$ in the direction of the unit vector $\mathbf{a}$. The resolved part of this derivative in the tangential direction $\mathbf{n}\times\mathbf{a}$, perpendicular to $\mathbf{a}$, is $(\mathbf{a}\cdot\nabla\mathbf{t})\cdot(\mathbf{n}\times\mathbf{a})$, which may also be written $\mathbf{a}\cdot(\nabla\mathbf{t}\times\mathbf{n})\cdot\mathbf{a}$. We shall call this the *swerve* of the family of curves* in the direction of $\mathbf{a}$. Denoting it by S we have

$$S=\mathbf{a}\cdot(\nabla\mathbf{t}\times\mathbf{n})\cdot\mathbf{a} \quad\text{.....................(18).}$$

If then we introduce the conic

$$\mathbf{r}\cdot(\nabla\mathbf{t}\times\mathbf{n})\cdot\mathbf{r}=1 \quad\text{.....................(19),}$$

whose centre is the point P, and whose plane is the tangent plane at P, it follows that *the swerve of the family of curves in any direction is equal to the inverse square of the radius of the conic* (19) *in that direction.*

For a tangential direction, inclined at an angle θ to that of $\mathbf{t}$, the value of the swerve is given by

$$S=(\mathbf{t}\cos\theta+\mathbf{t}'\sin\theta)\cdot\nabla\mathbf{t}\cdot(\mathbf{t}'\cos\theta-\mathbf{t}\sin\theta).$$

Since the unit vector $\mathbf{t}$ is perpendicular to its derivatives, the second term in the last factor contributes nothing to the value of the product. Consequently

$$\begin{aligned} S&=\cos\theta\,[\cos\theta\,(\mathbf{t}\cdot\nabla\mathbf{t}\cdot\mathbf{t}')+\sin\theta\,(\mathbf{t}'\cdot\nabla\mathbf{t}\cdot\mathbf{t}')]\\ &=\cos\theta\,(\sin\theta\operatorname{div}\mathbf{t}-\cos\theta\operatorname{div}\mathbf{t}') \quad\text{..................(20).}\end{aligned}$$

Thus the asymptotes of the conic (19), or the directions of zero swerve, are determined by

$$\cos\theta=0 \quad\text{and}\quad \tan\theta=\frac{\operatorname{div}\mathbf{t}'}{\operatorname{div}\mathbf{t}} \quad\text{............(21).}$$

The second of these directions coincides with that of an asymptote of the conic (9); the first is perpendicular to $\mathbf{t}$. Thus the axes of the conic (19) bisect the angles between those of the conic (9). From (20) it also follows that:

The sum of the swerves in two perpendicular directions at a point is invariant, and equal to the geodesic curvature of the family at that point.

* Neville uses the term "swerve" for a different function. See his *Multilinear Functions of Direction*, p. 26.

The value of this invariant also follows from that of the scalar of the dyadic $\nabla \mathbf{t} \times \mathbf{n}$. For

$$(\nabla \mathbf{t} \times \mathbf{n})_s = \left(\Sigma \frac{1}{E} \mathbf{r}_1 \ \mathbf{t}_1 \times \mathbf{n}\right)_s = \Sigma \frac{1}{E} \mathbf{r}_1 \cdot \mathbf{t}_1 \times \mathbf{n} = \mathbf{n} \cdot \operatorname{rot} \mathbf{t},$$

which is the geodesic curvature of the family of curves (Art. 8).

The swerve in the direction of $\mathbf{t}$ is equal to the geodesic curvature; that in the perpendicular direction is zero. The two directions of zero swerve will be perpendicular where $\operatorname{div} \mathbf{t}' = 0$. They will be coincident where $\operatorname{div} \mathbf{t} = 0$. Hence:

The locus of the points at which the two directions of zero swerve are coincident is the line of striction of the family of curves. The locus of the points at which these two directions are perpendicular is the line of normal curvature of the family.

FUNCTIONS OF DIRECTION ON A SURFACE

85. Mainardi-Codazzi relations. Gauss equation. When the parametric curves are orthogonal, the Mainardi-Codazzi relations and the Gauss characteristic equation (Vol. I, Chap. V) may be neatly and conveniently expressed in terms of the geometrical properties of those curves. With the usual notation let $\mathbf{a}$ and $\mathbf{b}$ denote the unit tangents to the curves $v =$ const. and $u =$ const. respectively. The normal curvatures κ_n, κ_n' in these directions are determined by

$$\left.\begin{aligned} \kappa_n &= \frac{L}{E} = -\mathbf{a} \cdot \nabla \mathbf{n} \cdot \mathbf{a} \\ \kappa_n' &= \frac{N}{G} = -\mathbf{b} \cdot \nabla \mathbf{n} \cdot \mathbf{b} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(22),$$

and the geodesic curvatures γ, γ' of the parametric curves by

$$\left.\begin{aligned} \gamma &= -\operatorname{div} \mathbf{b} = -\frac{E_2}{2E\sqrt{G}} \\ \gamma' &= \quad \operatorname{div} \mathbf{a} = \frac{G_1}{2G\sqrt{E}} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(23).$$

The torsion, τ, of the geodesic in the direction of $\mathbf{a}$ is

$$\tau = \frac{M}{\sqrt{EG}} = \mathbf{a} \cdot \operatorname{rot} \mathbf{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots(24),$$

and that of the geodesic in the direction of $\mathbf{b}$ is $-\tau$.

Since the parametric curves are orthogonal, the *Mainardi-Codazzi relations* reduce to

$$\left.\begin{aligned} L_2 - M_1 &= \tfrac{1}{2}E_2\left(\frac{L}{E}+\frac{N}{G}\right)+\tfrac{1}{2}M\left(\frac{G_1}{G}-\frac{E_1}{E}\right) \\ N_1 - M_2 &= \tfrac{1}{2}G_1\left(\frac{L}{E}+\frac{N}{G}\right)+\tfrac{1}{2}M\left(\frac{E_2}{E}-\frac{G_2}{G}\right) \end{aligned}\right\}.$$

From the first of these it follows that

$$\frac{1}{\sqrt{G}}\frac{\partial}{\partial v}\left(\frac{L}{E}\right)-\frac{1}{\sqrt{E}}\frac{\partial}{\partial u}\left(\frac{M}{\sqrt{EG}}\right)=\frac{M}{\sqrt{EG}}\frac{G_1}{G\sqrt{E}}+\left(\frac{N}{G}-\frac{L}{E}\right)\frac{E_2}{2E\sqrt{G}},$$

and we may write this, in terms of the above notation,

$$\mathbf{b}\cdot\nabla\kappa_n - \mathbf{a}\cdot\nabla\tau = 2\tau\gamma' + (\kappa_n - \kappa_n')\,\gamma \quad \text{........(25)}.$$

Similarly the second of the Mainardi-Codazzi relations may be reduced to the form

$$\mathbf{a}\cdot\nabla\kappa_n' - \mathbf{b}\cdot\nabla\tau = -2\tau\gamma + (\kappa_n - \kappa_n')\,\gamma' \quad \text{........(26)}.$$

We have already seen that the *Gauss characteristic equation* is equivalent to (Art. 13)

$$\begin{aligned} -K &= \operatorname{div}(\mathbf{a}\operatorname{div}\mathbf{a}+\mathbf{b}\operatorname{div}\mathbf{b}) \\ &= (\operatorname{div}\mathbf{a})^2 + (\operatorname{div}\mathbf{b})^2 + \mathbf{a}\cdot\nabla\operatorname{div}\mathbf{a} + \mathbf{b}\cdot\nabla\operatorname{div}\mathbf{b}. \end{aligned}$$

It may therefore be expressed in the form

$$\tau^2 - \kappa_n\kappa_n' = \gamma^2 + \gamma'^2 + \mathbf{a}\cdot\nabla\gamma' - \mathbf{b}\cdot\nabla\gamma \quad \text{........(27)}.$$

86. Functions of Laguerre and Darboux. Let $\dfrac{d}{ds}$ be used for the moment as the equivalent of $\mathbf{a}\cdot\nabla$, thus denoting differentiation in the direction of $\mathbf{a}$; and similarly let $\dfrac{d}{ds'}$ be used as the equivalent of $\mathbf{b}\cdot\nabla$, denoting differentiation in the direction of $\mathbf{b}$. Then, on differentiating the identity

$$-\kappa_n = \mathbf{a}\cdot\nabla\mathbf{n}\cdot\mathbf{a}$$

for the direction of $\mathbf{a}$, we have

$$-\frac{d\kappa_n}{ds}=\frac{d\mathbf{a}}{ds}\cdot\nabla\mathbf{n}\cdot\mathbf{a}+\mathbf{a}\cdot\left(\frac{d}{ds}\nabla\mathbf{n}\right)\cdot\mathbf{a}+\mathbf{a}\cdot\nabla\mathbf{n}\cdot\frac{d\mathbf{a}}{ds}.$$

Now, since $\nabla\mathbf{n}$ is symmetric, the first and the last terms in the second member are equal. Inserting the values of the derivatives

$$\mathbf{a}\cdot\nabla\mathbf{a} = \kappa_n\mathbf{n} + \gamma\mathbf{b}, \quad \mathbf{a}\cdot\nabla\mathbf{n} = -(\kappa_n\mathbf{a} + \tau\mathbf{b}),$$

we obtain a result which may be written

$$\mathbf{a}\cdot\nabla\kappa_n - 2\tau\gamma = -\mathbf{a}\cdot\left(\frac{d}{ds}\nabla\mathbf{n}\right)\cdot\mathbf{a} \quad \text{...........(28).}$$

Now the expression on the right depends only on the direction of $\mathbf{a}$, and not on the other properties of the curve $v=$ const.; for it does not involve the derivative of $\mathbf{a}$. It may be described as a function of the direction $\mathbf{a}$ on the surface. This property was first noticed by Laguerre; and the function is known as *Laguerre's function* for the direction of $\mathbf{a}$. We shall denote it by $\mathscr{L}$. Thus

$$\left.\begin{aligned}\mathscr{L} &= -\mathbf{a}\cdot\left(\frac{d}{ds}\nabla\mathbf{n}\right)\cdot\mathbf{a}\\ &= \mathbf{a}\cdot\nabla\kappa_n - 2\tau\gamma\end{aligned}\right\} \quad \text{.................(29).}$$

Laguerre's function for the direction of $\mathbf{b}$ is obtained on replacing κ_n by κ_n', γ by γ' and τ by $-\tau$. We shall denote it by $\mathscr{L}'$. Thus

$$\left.\begin{aligned}\mathscr{L}' &= -\mathbf{b}\cdot\left(\frac{d}{ds'}\nabla\mathbf{n}\right)\cdot\mathbf{b}\\ &= \mathbf{b}\cdot\nabla\kappa_n' + 2\tau\gamma'\end{aligned}\right\} \quad \text{.................(30).}$$

A similar function of direction was first observed by Darboux. Differentiating the identity

$$-\tau = \mathbf{a}\cdot\nabla\mathbf{n}\cdot\mathbf{b}$$

for the direction of $\mathbf{a}$, we have

$$-\frac{d\tau}{ds} = \frac{d\mathbf{a}}{ds}\cdot\nabla\mathbf{n}\cdot\mathbf{b} + \mathbf{a}\cdot\left(\frac{d}{ds}\nabla\mathbf{n}\right)\cdot\mathbf{b} + \mathbf{a}\cdot\nabla\mathbf{n}\cdot\frac{d\mathbf{b}}{ds}.$$

Now, since $\nabla\mathbf{n}$ is symmetric,

$$\nabla\mathbf{n}\cdot\mathbf{b} = \mathbf{b}\cdot\nabla\mathbf{n} = -(\tau\mathbf{a} + \kappa_n'\mathbf{b}),$$

while
$$\mathbf{a}\cdot\nabla\mathbf{b} = \tau\mathbf{n} - \gamma\mathbf{a}.$$

Inserting these values, and those of $\mathbf{a}\cdot\nabla\mathbf{a}$ and $\mathbf{a}\cdot\nabla\mathbf{n}$ given above, we obtain a result which may be expressed

$$\mathbf{a}\cdot\nabla\tau + (\kappa_n - \kappa_n')\gamma = -\mathbf{a}\cdot\left(\frac{d}{ds}\nabla\mathbf{n}\right)\cdot\mathbf{b} \quad \text{.........(31).}$$

Again the function on the right depends only on the direction of $\mathbf{a}$, for $\mathbf{b} = \mathbf{n}\times\mathbf{a}$. It is *Darboux's function* for the direction of $\mathbf{a}$, and will be denoted by $\mathscr{D}$. Thus

$$\left.\begin{aligned}\mathscr{D} &= -\mathbf{a}\cdot\left(\frac{d}{ds}\nabla\mathbf{n}\right)\cdot\mathbf{b}\\ &= \mathbf{a}\cdot\nabla\tau + (\kappa_n - \kappa_n')\gamma\end{aligned}\right\} \quad \text{...............(32).}$$

Darboux's function for the direction of **b** is obtained on replacing **a** by **b**, **b** by $-\mathbf{a}$, and so on. We shall denote it by $\mathscr{D}'$. Thus

$$\left.\begin{aligned}\mathscr{D}' &= \mathbf{b}\cdot\left(\frac{d}{ds'}\nabla\mathbf{n}\right)\cdot\mathbf{a}\\ &= -\mathbf{b}\cdot\nabla\tau-(\kappa_n-\kappa_n')\,\gamma'\end{aligned}\right\}\quad\text{..............(33).}$$

Laguerre lines on the surface are lines along which the Laguerre function for the direction of the curve vanishes identically. Similarly, *Darboux lines* are lines along which the Darboux function for the direction of the curve is zero.

From the above definitions, and the Mainardi-Codazzi relation (26), it follows at once that

$$\mathscr{L}-\mathscr{D}'=\mathbf{a}\cdot\nabla(\kappa_n+\kappa_n')=\mathbf{a}\cdot\nabla J \quad\text{...........(34),}$$

and similarly, in virtue of (25),

$$\mathscr{L}'+\mathscr{D}=\mathbf{b}\cdot\nabla(\kappa_n+\kappa_n')=\mathbf{b}\cdot\nabla J \quad\text{...........(35).}$$

These are both included in the statement:

The difference of the Laguerre function for any direction and the Darboux function for the perpendicular direction, obtained by a positive rotation of one right angle about the normal, is equal to the derivative of the first curvature in the former direction.

The expressions for $\mathscr{D}$ and $\mathscr{D}'$, given in (32) and (33), may be replaced by equivalent expressions. On differentiating the identity

$$-\kappa_n=\mathbf{a}\cdot\nabla\mathbf{n}\cdot\mathbf{a}$$

in the direction of **b**, we have

$$\begin{aligned}-\mathbf{b}\cdot\nabla\kappa_n &= 2\,(\mathbf{b}\cdot\nabla\mathbf{a})\cdot\nabla\mathbf{n}\cdot\mathbf{a}+\mathbf{a}\cdot\left(\frac{d}{ds'}\nabla\mathbf{n}\right)\cdot\mathbf{a}\\ &= -2\tau\gamma'+\mathbf{a}\cdot\left(\frac{d}{ds'}\nabla\mathbf{n}\right)\cdot\mathbf{a}.\end{aligned}$$

In virtue of (25) it follows that

$$-\mathbf{a}\cdot\left(\frac{d}{ds'}\nabla\mathbf{n}\right)\cdot\mathbf{a}=\mathbf{a}\cdot\nabla\tau+(\kappa_n-\kappa_n')\,\gamma=\mathscr{D}.$$

Consequently $\qquad \mathbf{a}\cdot\left(\dfrac{d}{ds'}\nabla\mathbf{n}\right)\cdot\mathbf{a}=\mathbf{a}\cdot\left(\dfrac{d}{ds}\nabla\mathbf{n}\right)\cdot\mathbf{b}.$

Thus the direction of the differentiation of $\nabla\mathbf{n}$ may be interchanged with that of the final vector in the product. But the first

and the last vectors in the product are also interchangeable; for $\nabla\mathbf{n}$ is symmetric, and consequently also $\frac{d}{ds}\nabla\mathbf{n}$. Thus

$$-\mathscr{D} = \mathbf{a}\cdot\left(\frac{d}{ds}\nabla\mathbf{n}\right)\cdot\mathbf{b} = \mathbf{b}\cdot\left(\frac{d}{ds}\nabla\mathbf{n}\right)\cdot\mathbf{a} = \mathbf{a}\cdot\left(\frac{d}{ds'}\nabla\mathbf{n}\right)\cdot\mathbf{a} \quad \text{......(36).}$$

Similarly, by differentiating the identity

$$-\kappa_n{}' = \mathbf{b}\cdot\nabla\mathbf{n}\cdot\mathbf{b}$$

in the direction of $\mathbf{a}$, we arrive at the equalities

$$\mathscr{D}' = \mathbf{b}\cdot\left(\frac{d}{ds'}\nabla\mathbf{n}\right)\cdot\mathbf{a} = \mathbf{a}\cdot\left(\frac{d}{ds'}\nabla\mathbf{n}\right)\cdot\mathbf{b} = \mathbf{b}\cdot\left(\frac{d}{ds}\nabla\mathbf{n}\right)\cdot\mathbf{b} \quad \text{......(37).}$$

***87. Codazzi function of three directions.** Consider the three unit surface vectors

$$\mathbf{f} = f\mathbf{a} + f'\mathbf{b}, \quad \mathbf{g} = g\mathbf{a} + g'\mathbf{b}, \quad \mathbf{h} = h\mathbf{a} + h'\mathbf{b} \quad \text{......(38).}$$

The derivative of $\nabla\mathbf{n}$ in the direction of the second is

$$g\frac{d}{ds}\nabla\mathbf{n} + g'\frac{d}{ds'}\nabla\mathbf{n}.$$

Forming the direct product of this dyadic with the other two vectors, and changing the sign of the result, we obtain the function

$$-(f\mathbf{a} + f'\mathbf{b})\cdot\left(g\frac{d}{ds}\nabla\mathbf{n} + g'\frac{d}{ds'}\nabla\mathbf{n}\right)\cdot(h\mathbf{a} + h'\mathbf{b}).$$

In virtue of (36) and (37) this product has the value

$$fgh\mathscr{L} + f'g'h'\mathscr{L}' + (f'gh + fg'h + fgh')\mathscr{D} - (fg'h' + f'gh' + f'g'h)\mathscr{D}'.$$

This expression is symmetrical with respect to the three vectors (38), that is to say, with respect to the three surface directions of which it is a function. Neville† calls it the *Codazzi function* for those directions. If we denote it by $\mathscr{C}(\mathbf{f\,g\,h})$, it follows from the symmetry of the function that

$$\mathscr{C}(\mathbf{f\,g\,h}) = \mathscr{C}(\mathbf{f\,h\,g}) = \mathscr{C}(\mathbf{h\,g\,f}) = \text{etc.}$$

Also, from the definition of the function and the results of Art. 86, we have the relations

$$\mathscr{C}(\mathbf{a\,a\,a}) = \mathscr{L}, \quad \mathscr{C}(\mathbf{b\,b\,b}) = \mathscr{L}',$$
$$\mathscr{C}(\mathbf{a\,a\,b}) = \mathscr{D}, \quad \mathscr{C}(\mathbf{a\,b\,b}) = -\mathscr{D}'.$$

† *Multilinear Functions of Direction*, pp. 54–59.

EXAMPLES VIII

1. From the definition of "tendency," verify that the direction of the vector $\mathbf{t}\,\mathrm{div}\,\mathbf{t}+\mathbf{t}'\,\mathrm{div}\,\mathbf{t}'$ is one of zero tendency.

2. Verify that, for a family of curves, the direction conjugate to that of $\mathbf{t}$ is one of zero moment.

If $\mathbf{s}$ is a vector in this conjugate direction,

$$\mathbf{s}\cdot\nabla\mathbf{n}\cdot\mathbf{t}=0 \quad\text{(i)}.$$

But, in virtue of the identity $\mathbf{t}\cdot\mathbf{n}=0$, we have

$$0=\nabla(\mathbf{t}\cdot\mathbf{n})=\nabla\mathbf{n}\cdot\mathbf{t}+\nabla\mathbf{t}\cdot\mathbf{n} \quad\text{(ii)}.$$

Consequently (i) may be written

$$\mathbf{s}\cdot\nabla\mathbf{t}\cdot\mathbf{n}=0,$$

or

$$\mathbf{s}\cdot(\nabla\mathbf{t}\times\mathbf{t})\cdot\mathbf{s}=0 \quad\text{(iii)},$$

since $\mathbf{t}\times\mathbf{s}$ is parallel to $\mathbf{n}$. From (iii) it follows that the direction of $\mathbf{s}$ is one of zero moment.

3. Considering two families of curves on a given surface, show that the moment of the first in the direction of the second is the negative of the moment of the second in the direction of the first. Deduce that the sum of the moments of each in the direction perpendicular to the other is equal to the sum of their total moments.

Let $\mathbf{c}$ and $\mathbf{d}$ be the unit tangents for the two families, cutting at an angle θ. From (ii) of Ex. 2, and the symmetry of $\nabla\mathbf{n}$, we have immediately

$$\mathbf{c}\cdot\nabla\mathbf{d}\cdot\mathbf{n}=\mathbf{d}\cdot\nabla\mathbf{c}\cdot\mathbf{n},$$

and hence, since

$$\mathbf{c}\times\mathbf{d}=\mathbf{n}\sin\theta,$$

$$\mathbf{c}\cdot(\nabla\mathbf{d}\times\mathbf{d})\cdot\mathbf{c}=-\mathbf{d}\cdot(\nabla\mathbf{c}\times\mathbf{c})\cdot\mathbf{d} \quad\text{(i)},$$

which is the first of the required results. From this, and the fact that the total moment of a family is the sum of the moments for two perpendicular directions, the second result follows.

4. With the notation of Ex. 3, show that the moment of the family $\mathbf{c}$ in the direction of $\mathbf{d}$ has the value

$$-(\mathbf{c}\cdot\nabla\mathbf{n}\cdot\mathbf{d})\sin\theta.$$

5. Defining the "screw curvature" of a curve as the arc-rate of change of the unit principal normal, show that the screw curvature of the geodesic in any direction $\mathbf{t}$ on the surface is parallel to the normal to the indicatrix at the end of the diameter in the direction of $\mathbf{t}$. Also show that the square of this screw curvature has the value $\kappa_a^2\cos^2\theta+\kappa_b^2\sin^2\theta$, where κ_a, κ_b are the principal curvatures, and θ is the inclination of $\mathbf{t}$ to the principal direction corresponding to κ_a.

6. Show that, for the orthogonal system consisting of the lines of curvature,

$$\mathbf{b}\cdot\nabla\kappa_n=(\kappa_n-\kappa_n')\,\gamma=-(\kappa_n-\kappa_n')\,\mathrm{div}\,\mathbf{b},$$

$$\mathbf{a}\cdot\nabla\kappa_n'=(\kappa_n-\kappa_n')\,\gamma'=\quad(\kappa_n-\kappa_n')\,\mathrm{div}\,\mathbf{a}.$$

7. Prove the equalities (37) as in the case of (36).

8. By differentiating the identity

$$\kappa_n + \kappa_n' = -\mathbf{a} \cdot \nabla \mathbf{n} \cdot \mathbf{a} - \mathbf{b} \cdot \nabla \mathbf{n} \cdot \mathbf{b}$$

in the direction of $\mathbf{a}$, and using the relations (37), verify the identity (34). Similarly verify (35).

9. Show that, if $\mathbf{a}$ and $\mathbf{b}$ are in the principal directions,

$$\mathscr{C}(\mathbf{a}\,\mathbf{a}\,\mathbf{a}) = \mathbf{a} \cdot \nabla \kappa_n, \quad \mathscr{C}(\mathbf{b}\,\mathbf{b}\,\mathbf{b}) = \mathbf{b} \cdot \nabla \kappa_n',$$

$$\mathscr{C}(\mathbf{a}\,\mathbf{a}\,\mathbf{b}) = \mathbf{b} \cdot \nabla \kappa_n, \quad \mathscr{C}(\mathbf{a}\,\mathbf{b}\,\mathbf{b}) = \mathbf{a} \cdot \nabla \kappa_n'.$$

10. If the direction of $\mathbf{a}$ is asymptotic, show that

$$\mathscr{C}(\mathbf{a}\,\mathbf{a}\,\mathbf{a}) = -2\tau\gamma.$$

11. If $\dfrac{d}{ds_1}, \dfrac{d}{ds_2}, \dfrac{d}{ds_3}$ denote differentiation in the directions of the *unit* surface vectors $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$ respectively, show that

$$\mathscr{C}(\mathbf{c}\,\mathbf{d}\,\mathbf{e}) = \frac{d\mathbf{c}}{ds_2} \cdot \frac{d\mathbf{n}}{ds_3} + \frac{d\mathbf{e}}{ds_2} \cdot \frac{d\mathbf{n}}{ds_1} - \frac{d}{ds_2}\left(\mathbf{e} \cdot \frac{d\mathbf{n}}{ds_1}\right).$$

CHAPTER IX

LEVI-CIVITA'S PARALLEL DISPLACEMENTS. TCHEBYCHEF SYSTEMS

88. Levi-Civita's conception of parallelism*. In this chapter we shall give a brief account of Levi-Civita's theory of parallelism of directions with respect to a surface, and of parallel displacement of a vector along a given curve on the surface. The results obtained will then be applied to investigate some properties of Tchebychef systems of curves on a surface.

Starting with the plane as the simplest of surfaces, we say that two surface directions at points P and P' are parallel, when they are parallel in the Euclidean sense, making equal angles in the same sense with the direction PP'. Next consider a developable surface. Let $\mathbf{u}$ and $\mathbf{u}'$ be surface vectors at the points P and P' of this surface. The directions of these vectors are said to be parallel with respect to the developable if they become parallel in the above sense when the surface is developed into a plane. Since geodesics on a developable become straight lines when the surface is developed into a plane, it follows that the directions of $\mathbf{u}$ and $\mathbf{u}'$ are parallel with respect to the developable when they are equally inclined, in the same sense, to the geodesic joining P and P'.

We might generalise this result by defining two surface directions $\mathbf{u}$ and $\mathbf{u}'$, at points P and P' respectively on any surface, as parallel with respect to that surface when they are equally inclined to the geodesic joining P and P'. This, however, is less general than the definition adopted by Levi-Civita, who makes the parallelism of two surface directions at P and P' to depend on the curve joining these points. Let C be any curve drawn on the surface through P and P', and let Σ be a developable surface touching the given surface S along the curve C. Then $\mathbf{u}$ and $\mathbf{u}'$ are also surface vectors for the developable. Levi-Civita defines their directions as parallel with respect to the surface S and the connecting

* See Levi-Civita, *The Absolute Differential Calculus*, pp. 101–119 and 193–198, also *Palermo Rendiconti*, Vol. 42 (1917), pp. 173–204.

curve C, if they become parallel in the Euclidean sense when Σ is developed into a plane.

Let the point P move along the curve C, and let the surface vector $\mathbf{u}$ vary with the position of P so that, for every position of P, it is parallel to its original direction in the above sense, with respect to the curve C and the surface S. Then $\mathbf{u}$ is said to undergo a *parallel displacement* along the curve C. When P reaches the point P', $\mathbf{u}$ has the direction of $\mathbf{u}'$. Every value of $\mathbf{u}$ along C is tangential to the surface Σ; and, when Σ is developed into a plane, the various directions of $\mathbf{u}$ along C become parallel directions in that plane.

It follows immediately from the above definition that *angles are unchanged by parallel displacement*; that is to say, if the directions $\mathbf{u}$, $\mathbf{v}$ at P are parallel respectively to the directions $\mathbf{u}'$, $\mathbf{v}'$ at P', the angle between the former pair is equal to the angle between the latter, whatever the curve C of displacement. For these angles are unchanged when Σ is developed into a plane; and since in the plane $\mathbf{u}$ and $\mathbf{u}'$ are parallel in the Euclidean sense, and also $\mathbf{v}$ and $\mathbf{v}'$, the result follows.

It is also clear that, when the curve C of displacement is a geodesic on S, a vector $\mathbf{u}$ which undergoes a parallel displacement along C is inclined at a constant angle to this curve. For C is also a geodesic on Σ, since the normal to S at each point of C is also the normal to Σ. Hence, when Σ is developed into a plane, C becomes a straight line, and the vector $\mathbf{u}$ which undergoes a parallel displacement cuts this line at a constant angle. Conversely, any surface direction at points of a geodesic, which is inclined at a constant angle to the curve, undergoes a parallel displacement along the geodesic. In particular the direction of the geodesic itself undergoes a parallel displacement along the curve. For this reason geodesics are said to be *auto-parallel* curves.

89. Condition for a parallel displacement. Let the vector $\mathbf{u}$, which undergoes a parallel displacement along C, be of constant length (say unity). Let $\mathbf{u}$ be its value at the point P, and $\mathbf{u} + \delta\mathbf{u}$ that at an adjacent point Q on the curve, s being the arc-length of C and δs the infinitesimal arc-length PQ. Now when Σ is developed into a plane, the tangent plane at P becomes identical

with that at Q. Thus the change $\delta\mathbf{u}$ in the vector, due to the displacement δs along the curve, is that due to a rotation of the tangent plane at P about an axis through P in the direction conjugate to that of the curve. Denoting the arc-rate of rotation

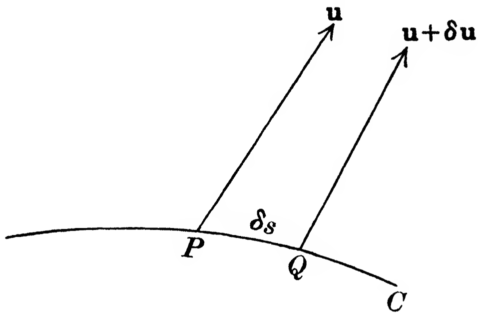

Fig. 7.

of the tangent plane, as the point of contact moves along the curve, by the tangential vector $\mathbf{A}$, we have to the first order

$$\delta\mathbf{u} = \delta s\mathbf{A} \times \mathbf{u}.$$

Consequently, on taking limiting values as Q tends to coincidence with P,

$$\frac{d\mathbf{u}}{ds} = \mathbf{A} \times \mathbf{u} \quad \text{.........................(1).}$$

Now both $\mathbf{A}$ and $\mathbf{u}$ are surface vectors, so that $\mathbf{A} \times \mathbf{u}$ is normal to the surface. Hence:

If a vector of constant length undergoes a parallel displacement along a curve on the surface, its rate of change along the curve is normal to the surface.

We have already seen that the vector $\mathbf{A}$ has the direction conjugate to that of C; and, if C is regarded as a member of a family of curves with unit tangent $\mathbf{t}$, $\mathbf{A}$ is the tangential part of rot $\mathbf{t}$ (Art. 9). Thus

$$\begin{aligned}\frac{d\mathbf{u}}{ds} &= \{(\mathbf{n} \times \operatorname{rot}\mathbf{t}) \times \mathbf{n}\} \times \mathbf{u} \\ &= \{\mathbf{u} \cdot (\mathbf{n} \times \operatorname{rot}\mathbf{t})\}\, \mathbf{n} \quad \text{.....................(2).}\end{aligned}$$

From the above theorem we may easily deduce an important formula giving the arc-rate of change of the inclination of $\mathbf{u}$ to the

curve, due to the parallel displacement. As above, we may regard C as a member of a family T of curves with unit tangent $\mathbf{t}$ and arc-length s. Let $\mathbf{u}$ be a *unit* vector cutting C at a variable angle θ, so that θ is the angle of rotation from $\mathbf{t}$ to $\mathbf{u}$ in the positive sense about the normal. If then $\mathbf{t}'$ is the unit surface vector $\mathbf{n} \times \mathbf{t}$, perpendicular to $\mathbf{t}$, we may write

$$\mathbf{u} = \mathbf{t} \cos \theta + \mathbf{t}' \sin \theta.$$

Then, since $\dfrac{d\mathbf{u}}{ds}$ is normal to the surface, it follows that

$$0 = \mathbf{t} \cdot \frac{d\mathbf{u}}{ds} = \sin \theta \left(\mathbf{t} \cdot \frac{d\mathbf{t}'}{ds} - \frac{d\theta}{ds} \right),$$

and

$$0 = \mathbf{t}' \cdot \frac{d\mathbf{u}}{ds} = \cos \theta \left(\mathbf{t}' \cdot \frac{d\mathbf{t}}{ds} + \frac{d\theta}{ds} \right).$$

Since then

$$\mathbf{t}' \cdot \frac{d\mathbf{t}}{ds} = - \mathbf{t} \cdot \frac{d\mathbf{t}'}{ds} = \gamma,$$

where γ is the geodesic curvature* of C, the above relations show that

$$\frac{d\theta}{ds} = - \gamma \qquad \text{(3)}.$$

This is Levi-Civita's formula for the arc-rate of change of θ along C. Thus:

If a vector $\mathbf{u}$ *undergoes a parallel displacement along a curve C, the arc-rate of increase of the inclination of* $\mathbf{u}$ *to the curve is the negative of the geodesic curvature of C.*

It is then evident that, if two vectors undergo parallel displacements along the same curve, they are inclined to each other at a constant angle; or, angles are unchanged by a parallel displacement. From (3) it is also obvious that, if the curve C is a geodesic, the direction of $\mathbf{u}$ is inclined at a constant angle to that of the curve.

Since the derivative of $\mathbf{u}$ along C is normal to the surface, its magnitude is given by

$$\mathbf{n} \cdot \frac{d\mathbf{u}}{ds} = \mathbf{n} \cdot \left(\frac{d\mathbf{t}}{ds} \cos \theta + \frac{d\mathbf{t}'}{ds} \sin \theta \right)$$
$$= \kappa_n \cos \theta + \tau \sin \theta,$$

* The above proof was given by the author, *Bull. Amer. Math. Soc.*, Vol. 34 (1928), pp. 585–586.

where κ_n is the normal curvature of the surface in the direction of $\mathbf{t}$, and τ is the torsion of the geodesic tangent to C. This magnitude is clearly the resolved part of the vector $\kappa_n \mathbf{t} + \tau \mathbf{t}'$ in the direction of $\mathbf{u}$. The equation (2) is easily shown to be in agreement with this result.

90. Two theorems on parallel displacement. Consider a family of curves A with unit tangent $\mathbf{a}$, and their orthogonal trajectories B with unit tangent $\mathbf{b}$, such that $\mathbf{n} = \mathbf{a} \times \mathbf{b}$. Let a third family T cut the first at a variable angle $-\theta$, so that the unit tangent $\mathbf{t}$ to this family is given by

$$\mathbf{t} = \mathbf{a} \cos \theta - \mathbf{b} \sin \theta.$$

Then the geodesic curvature of a curve of the family T has the value γ, where

$$\begin{aligned} -\gamma &= \operatorname{div}(\mathbf{a} \sin \theta + \mathbf{b} \cos \theta) \\ &= \sin \theta \operatorname{div} \mathbf{a} + \cos \theta \operatorname{div} \mathbf{b} + \mathbf{t} \cdot \nabla \theta \quad \text{.........(4).} \end{aligned}$$

If now the vector $\mathbf{a}$ is such that it suffers a parallel displacement along a curve of the family T, $\mathbf{t} \cdot \nabla \theta = -\gamma$ along this curve, and consequently

$$\sin \theta \operatorname{div} \mathbf{a} + \cos \theta \operatorname{div} \mathbf{b} = 0 \quad \text{..............(5).}$$

From this result two theorems may be deduced. First suppose that A is a family of parallels. Then div $\mathbf{a}$ vanishes identically; and the last equation shows that, along the curve T, either $\cos \theta = 0$ or div $\mathbf{b} = 0$. The second of these alternatives can be satisfied only where the geodesic curvature of the parallel curves is zero, that is to say, on the line of normal curvature of the family. Hence the theorem*:

If the tangent to a family of parallels undergoes a parallel displacement along a transversal, either this transversal is an orthogonal trajectory (and therefore a geodesic), or else it is a line of normal curvature of the family of parallels.

Secondly let the family A be one of geodesics. Then div $\mathbf{b}$ vanishes identically, and (5) shows that $\sin \theta \operatorname{div} \mathbf{a} = 0$ along the transversal T. Since then $\sin \theta$ is not zero, this requires div $\mathbf{a} = 0$,

* Given by the author in the *Bull. Amer. Math. Soc.*, loc. cit., p. 587.

which holds only at the line of striction of the family of geodesics. Hence the theorem*:

If the tangent to a family of geodesics on a surface undergoes a parallel displacement along a transversal, this transversal is a line of striction of the family of geodesics.

91. Parallel displacement round a closed curve. Let C be a closed curve on the surface S, described in the sense which is positive for rotation about the normal. Suppose that, as the point P moves round the curve in the positive sense, the surface vector $\mathbf{u}$ undergoes a parallel displacement. Let $\mathbf{u}'$ be the value of the vector at any point, and $\mathbf{u}''$ its value at the same point after P has described the curve once in the positive sense. Then the angle of inclination, measured from $\mathbf{u}'$ to $\mathbf{u}''$ is called the *angle of parallelism* for the curve C. We shall prove that it is independent of the starting point, and of the vector $\mathbf{u}$.

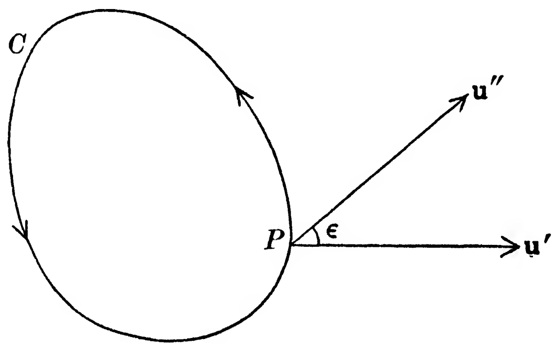

Fig. 8.

The total angular rotation of $\mathbf{u}$ is its rotation relative to the tangent to C, increased by the rotation of that tangent. Now, since the arc-rate of rotation of $\mathbf{u}$ relative to the tangent has the value $-\gamma$, the total rotation relative to the tangent is equal to $-\int_0 \gamma ds$. Also, for a simple closed curve, the rotation of the tangent during one description of the curve is 2π. Consequently the total rotation of $\mathbf{u}$ is $2\pi - \int_0 \gamma ds$. But, in virtue of the Gauss-Bonnet formula

* Bianchi, *Geometria Differenziale*, Vol. 2, p. 802.

(Art. 18), this expression is equal to the surface integral of the second curvature of the surface taken over the region enclosed by C. Consequently the angle of parallelism ϵ for the curve C is given by

$$\epsilon = \iint K dS \qquad \text{(6)},$$

and is therefore independent of $\mathbf{u}$ and of the starting point.

Let C be an infinitesimal closed curve surrounding the point Q, ϵ the angle of parallelism for the curve, and δS the area of the enclosed region. Then it follows from (6) that the second curvature of the surface at Q is the limiting value of the quotient of the angle of parallelism for the curve by the enclosed area, as the curve C converges to the point Q; or

$$K = \mathrm{Lt}\,\frac{\epsilon}{\delta S} \qquad \text{(7)}.$$

TCHEBYCHEF SYSTEMS.

92. Tchebychef nets. Suppose that the parameters u, v on a given surface are such that, in the expression for the linear element

$$ds^2 = E du^2 + 2F du dv + G dv^2,$$

E is a function of u only, and G a function of v only, so that

$$E_2 = 0, \quad G_1 = 0 \qquad \text{(8)}.$$

Then the parametric curves are said to form a *Tchebychef system* or net on the given surface. The name is given in honour of Tchebychef, who was the first to study such systems. If now we put

$$\sqrt{E} du = d\alpha, \quad \sqrt{G} dv = d\beta,$$

the linear element is expressible in the form

$$ds^2 = d\alpha^2 + 2\cos\omega\, d\alpha d\beta + d\beta^2 \qquad \text{(9)},$$

where ω is the angle between the parametric curves. In terms of these parameters the second curvature of the surface, in virtue of the Gauss characteristic equation, is given by

$$K = -\frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial\alpha\partial\beta}.$$

Now we have seen that, in the identity (Vol. I, Art. 41)

$$\frac{\partial^2 \mathbf{r}}{\partial u \partial v} = M\mathbf{n} + m\mathbf{r}_1 + \mu\mathbf{r}_2,$$

the coefficients m, μ are given by

$$m = \tfrac{1}{2}H^{-2}(GE_2 - FG_1), \quad \mu = \tfrac{1}{2}H^{-2}(EG_1 - FE_2).$$

If the relations (8) hold, these coefficients vanish identically. Thus when the parametric curves form a Tchebychef system,

$$\mathbf{r}_{12} = M\mathbf{n} \quad \text{.........................(10).}$$

Conversely, when $\mathbf{r}_{12}$ is everywhere normal to the surface, m and μ both vanish identically and the relations (8) must hold. Consequently the parametric curves form a Tchebychef net.

The unit tangent to the curve $v =$ const. is the vector $\mathbf{a} = \mathbf{r}_1/\sqrt{E}$; and its derivative in the direction of the curve $u =$ const. is

$$\begin{aligned}\frac{1}{\sqrt{G}}\frac{\partial}{\partial v}\left(\frac{\mathbf{r}_1}{\sqrt{E}}\right) &= \frac{1}{\sqrt{G}}\left(\frac{\mathbf{r}_{12}}{\sqrt{E}} - \frac{E_2\mathbf{r}_1}{2E\sqrt{E}}\right)\\ &= \frac{1}{\sqrt{EG}}\left(M\mathbf{n} + m\mathbf{r}_1 + \mu\mathbf{r}_2 - \frac{E_2}{2E}\mathbf{r}_1\right).\end{aligned}$$

When the relations (8) hold, the coefficients of $\mathbf{r}_1$ and $\mathbf{r}_2$ in this expression both vanish, and the derivative of $\mathbf{a}$ is normal to the surface. Thus the vector $\mathbf{a}$ undergoes a parallel displacement along the curve $u =$ const. Similarly it may be shown that the tangent to the curve $u =$ const. suffers a parallel displacement along the curve $v =$ const. Hence the theorem:

The tangent to either family of a Tchebychef system undergoes a parallel displacement along each curve of the other family.

Conversely, if this relation holds for the tangents to the coordinate curves, the coefficients m and μ must vanish identically. This requires $E_2 = G_1 = 0$, showing that the coordinate curves form a Tchebychef system.

An illustration of a Tchebychef net is afforded by the coordinate curves on a *surface of translation*. On such a surface the position vector of the current point is expressible in the form

$$\mathbf{r} = \mathbf{U} + \mathbf{V} \quad \text{.......................(11),}$$

where $\mathbf{U}$ is a function of u only, and $\mathbf{V}$ a function of v only. The surface may be generated by imposing on the curve $\mathbf{r} = \mathbf{U}(u)$ a translation in which each point describes a curve congruent with the curve $\mathbf{r} = \mathbf{V}(v)$, or by a translation of the latter curve in which each of its points describes a curve congruent with the former. This is the reason for the name "surface of translation." It follows

from (11) that $\mathbf{r}_{12}$ vanishes identically, so that $m = \mu = 0$ at all points, and consequently $E_2 = G_1 = 0$. Thus the coordinate curves form a Tchebychef system. They also form a *conjugate system* since

$$M = \mathbf{n} \cdot \mathbf{r}_{12} = 0,$$

identically.

93. Lines of striction and normal curvature. Let the curves of a Tchebychef system be taken as parametric curves, and let the coordinates be chosen so that the square of the linear element has the form

$$ds^2 = du^2 + 2 \cos \theta \, du \, dv + dv^2.$$

Then, since $E = G = 1$ and $F = \cos \theta$, we have $H = \sin \theta$. If $\mathbf{u}$ and $\mathbf{v}$ are the unit tangents to the parametric curves $v = \text{const.}$ and $u = \text{const.}$ respectively,

$$\operatorname{div} \mathbf{u} = \operatorname{div} \mathbf{r}_1 = \frac{1}{H} \frac{\partial H}{\partial u} = \cot \theta \frac{\partial \theta}{\partial u} \quad \text{.............(12)},$$

and similarly

$$\operatorname{div} \mathbf{v} = \cot \theta \frac{\partial \theta}{\partial v} \quad \text{.....................(13)}.$$

Let γ and γ' be the geodesic curvatures of the curves $v = \text{const.}$ and $u = \text{const.}$ respectively. Then, since the tangent to either family undergoes a parallel displacement along each curve of the other,

$$\frac{\partial \theta}{\partial u} = -\gamma, \quad \frac{\partial \theta}{\partial v} = \gamma' \quad \text{.....................(14)}.$$

Consequently (12) and (13) may be written

$$\operatorname{div} \mathbf{u} = -\gamma \cot \theta, \quad \operatorname{div} \mathbf{v} = \gamma' \cot \theta \quad \text{............(15)}.$$

Now $\operatorname{div} \mathbf{u} = 0$ is the line of striction of the family $v = \text{const.}$, and $\gamma = 0$ is its line of normal curvature. Similarly for the family $u = \text{const.}$ Thus*:

The line of striction of either family of a Tchebychef system consists of its line of normal curvature and the locus of points at which the two families cut orthogonally.

It follows that a line of normal curvature of either family is also a line of striction of that family.

* The substance of Arts. 93–94 was given by the author in a paper "On Levi-Civita's theory of Parallelism," *Bull. Amer. Math. Soc.*, Vol. 34 (1928), pp. 585–590.

If the two families of a Tchebychef net cut at a constant angle, it is evident from (12) and (13) that div $\mathbf{u}$ and div $\mathbf{v}$ vanish identically. Thus the two families are families of parallel curves, and in virtue of (14) they are also geodesics. Consequently the surface is developable (Art. 17), and we have the theorem:

If the two families of a Tchebychef system cut at a constant angle, each family is a family of parallel geodesics and the surface is developable.

94. Further properties of a Tchebychef system. If $\mathbf{u}$ and $\mathbf{v}$ are the unit tangents to the two families U and V respectively of a Tchebychef net, the derivative of $\mathbf{u}$ in the direction of $\mathbf{v}$ is normal to the surface, and is therefore perpendicular to $\mathbf{v}$. Consequently $\mathbf{v} \cdot \nabla \mathbf{u} \cdot \mathbf{v} = 0$; and similarly $\mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{u} = 0$. Conversely, if these relations hold, both the derivatives $\mathbf{v} \cdot \nabla \mathbf{u}$ and $\mathbf{u} \cdot \nabla \mathbf{v}$ are normal to the surface, and we have the theorem:

Necessary and sufficient conditions that the two families of curves with unit tangents $\mathbf{u}$, $\mathbf{v}$ *may constitute a Tchebychef system are expressed by the equations*

$$\mathbf{v} \cdot \nabla \mathbf{u} \cdot \mathbf{v} = 0, \quad \mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{u} = 0 \qquad \text{..............(16).}$$

That is to say, at any point P on the surface, the tendency of each family in the direction of the other must be zero. If now we introduce the conics

$$\mathbf{r} \cdot \nabla \mathbf{u} \cdot \mathbf{r} = 1 \qquad \text{......................(17),}$$

and

$$\mathbf{r} \cdot \nabla \mathbf{v} \cdot \mathbf{r} = 1 \qquad \text{......................(18),}$$

whose centres are at P, the origin of the vector $\mathbf{r}$, and whose plane is the tangent plane at P, it follows from (16) that $\mathbf{v}$ is in the direction of one of the asymptotes of (17). But the other asymptote has the direction of $\mathbf{u}$ (Art. 82); so that the tangents to the two curves at P are the asymptotes of (17). The same is true for the conic (18), showing that the two conics are similar. The tendency of the family U in any direction at P is equal to the inverse square of the radius of the conic (17) in that direction. Thus the ratio of the tendencies of the two families in any direction at P is invariant. But the sum of the tendencies of a family in two directions at right angles is equal to the divergence of the family. Hence:

The ratio of the tendencies of the two families of a Tchebychef

system in any direction at a given point is invariant, and equal to the ratio of their divergences at that point.

When the asymptotes of the conics (17) and (18) are at right angles, the divergence of each family is zero. Consequently, as we have already seen, the locus of the points at which the two families of curves cut orthogonally is on the line of striction of each family.

Again, since $\mathbf{v}\cdot\nabla\mathbf{u}$ is normal to the surface, its resolved part in the direction of $\mathbf{n}\times\mathbf{v}$ is zero. Consequently

$$\mathbf{v}\cdot(\nabla\mathbf{u}\times\mathbf{n})\cdot\mathbf{v}=0 \qquad (19),$$

and similarly

$$\mathbf{u}\cdot(\nabla\mathbf{v}\times\mathbf{n})\cdot\mathbf{u}=0 \qquad (20).$$

These equations express that the swerve of each family in the direction of the other is zero. Thus $\mathbf{v}$ has the direction of one of the asymptotes of the conic

$$\mathbf{r}\cdot(\nabla\mathbf{u}\times\mathbf{n})\cdot\mathbf{r}=1 \qquad (21).$$

But the other asymptote is perpendicular to $\mathbf{u}$; and the inclination of the asymptotes is therefore $\frac{\pi}{2}-\theta$. Similarly the asymptotes of the conic

$$\mathbf{r}\cdot(\nabla\mathbf{v}\times\mathbf{n})\cdot\mathbf{r}=1 \qquad (22)$$

are parallel to $\mathbf{u}$ and perpendicular to $\mathbf{v}$. These two conics are therefore similar, the asymptotes of each being perpendicular to those of the other. And, since the swerve of the family U in any direction is equal to the inverse square of the radius of the conic (21) in that direction, it follows that the ratio of the swerves of the two families in any two perpendicular directions at a point is invariant. But the sum of the swerves of a family in two perpendicular directions at a point is equal to the geodesic curvature of the family at that point (Art. 84). Hence the theorem:

The ratio of the swerves of the two families of a Tchebychef system in any two perpendicular directions at a point is invariant, and equal to the ratio of their geodesic curvatures at that point.

In virtue of (15) this ratio is the negative of the ratio of the divergences of the two families.

EXAMPLES IX

1. If a surface vector $\mathbf{u}$, of constant magnitude, undergoes a parallel displacement along a curve whose unit tangent is $\mathbf{t}$, then

$$\mathbf{t}\cdot\nabla\mathbf{u}=(\mathbf{t}\cdot\nabla\mathbf{u}\cdot\mathbf{n})\,\mathbf{n}=-(\mathbf{t}\cdot\nabla\mathbf{n}\cdot\mathbf{u})\,\mathbf{n}.$$

2. A surface vector of constant length, which undergoes a parallel displacement along a given curve on a surface, has a stationary value when its direction is conjugate to that of the curve.

3. With the notation of Arts. 93, 94 for a Tchebychef system, show that

$$\cos\theta\,\nabla\theta=\mathbf{n}\times(\mathbf{u}\operatorname{div}\mathbf{v}-\mathbf{v}\operatorname{div}\mathbf{u}),$$

and that
$$\gamma\operatorname{div}\mathbf{v}+\gamma'\operatorname{div}\mathbf{u}=0.$$

4. With the same notation, show that the unit vector in the direction of the bisector of the angle between the curves of a Tchebychef net is $\frac{1}{2}(\mathbf{u}+\mathbf{v})\sec\frac{1}{2}\theta$, and hence that the rate of increase of θ in this direction is

$$\frac{1}{2}\left(\frac{\partial\theta}{\partial u}+\frac{\partial\theta}{\partial v}\right)\sec\tfrac{1}{2}\theta=\tfrac{1}{2}(\gamma'-\gamma)\sec\tfrac{1}{2}\theta.$$

5. Show that, when the curves bisecting the angles of a Tchebychef system are taken as coordinate lines, the linear element of the surface is expressible in the form

$$ds^2=\cos^2\frac{\omega}{2}\,du^2+\sin^2\frac{\omega}{2}\,dv^2.$$

6. Prove Lie's theorem that the developable enveloping a surface of translation along a generating curve is a cylinder.

7. Show that the locus of the point which divides in a constant ratio the joins of points on two given curves, or all the chords of one curve, is a surface of translation. In the latter case the curve is an asymptotic line of the surface.

CHAPTER X

REPRESENTATION OF SURFACES. CONICAL PROJECTION

95. Linear transformation. Extension ratio. In conformal and spherical representations we have already met examples of the representation of one surface on another (Vol. I, Chap. IX). We shall now consider more generally the representation associated with a linear transformation.

Let $\mathbf{r}$ be the position vector of the current point P on a given surface S, and Ω a dyadic point-function on that surface. Consider another surface S' on which the point P' corresponding to P has a position vector $\mathbf{r}'$ defined by

$$\mathbf{r}' = \Omega \cdot \mathbf{r} \quad \text{.........................(1).}$$

Let Q, Q' be another pair of corresponding points adjacent to P, P' respectively; and let $d\mathbf{r}$, $d\mathbf{r}'$ denote the elementary vectors PQ, $P'Q'$. Then from (1) it follows that

$$\begin{aligned} d\mathbf{r}' &= d\Omega \cdot \mathbf{r} + \Omega \cdot d\mathbf{r} \\ &= (d\mathbf{r} \cdot \nabla\Omega) \cdot \mathbf{r} + d\mathbf{r} \cdot \Omega_c \\ &= d\mathbf{r} \cdot (\nabla\Omega \cdot \mathbf{r} + \Omega_c). \end{aligned}$$

This relation may be expressed in the form

$$d\mathbf{r}' = \Phi \cdot d\mathbf{r} \quad \text{.........................(2),}$$

where Φ is a dyadic point-function on S, and is independent of the direction of $d\mathbf{r}$. Thus the vector element $d\mathbf{r}'$ on S' is a linear vector function of the corresponding vector element $d\mathbf{r}$ on S, just as $\mathbf{r}'$ is a linear vector function of $\mathbf{r}$ given by (1). We propose to consider representations of a surface S on another S', characterised by a correspondence of vector elements of the form (2); and the results will be expressed in terms of the dyadic Φ.

Let ds and ds' be the lengths of the corresponding linear elements $d\mathbf{r}$ and $d\mathbf{r}'$ respectively. Then

$$(ds')^2 = (d\mathbf{r}')^2 = (d\mathbf{r} \cdot \Phi_c) \cdot (\Phi \cdot d\mathbf{r}).$$

Hence, if $\mathbf{t}$ is the unit vector in the direction of the element $d\mathbf{r}$, we have

$$\left(\frac{ds'}{ds}\right)^2 = \mathbf{t} \cdot \Phi_c \cdot \Phi \cdot \mathbf{t} \quad \text{.....................(3)}.$$

The quotient ds'/ds will be called the *extension ratio* in the direction $\mathbf{t}$ due to the transformation, and will be denoted by ϵ. Thus

$$\epsilon^2 = \mathbf{t} \cdot \Phi_c \cdot \Phi \cdot \mathbf{t} \quad \text{.......................(4)}.$$

If Φ is a complete* dyadic, and Φ^{-1} is its reciprocal in the sense of Art. 69, it follows from (2) that

$$d\mathbf{r} = \Phi^{-1} \cdot d\mathbf{r}',$$

and therefore $$ds^2 = d\mathbf{r}' \cdot \Phi_c^{-1} \cdot \Phi^{-1} \cdot d\mathbf{r}' \quad \text{.................(5)}.$$

Consequently, if $\mathbf{t}'$ is the unit surface vector at P' in the direction of $d\mathbf{r}'$,

$$\epsilon^{-2} = \mathbf{t}' \cdot \Phi_c^{-1} \cdot \Phi^{-1} \cdot \mathbf{t}' \quad \text{.................(5')}.$$

The corresponding unit surface vectors $\mathbf{t}$, $\mathbf{t}'$ at P, P' respectively are connected by the relation

$$\epsilon\mathbf{t}' = \Phi \cdot \mathbf{t}.$$

Consequently the angle ϕ' between the surface directions at P' which correspond to those of the unit surface vectors $\mathbf{s}$, $\mathbf{t}$ at P is given by

$$\epsilon_1 \epsilon_2 \cos \phi' = \mathbf{t} \cdot \Phi_c \cdot \Phi \cdot \mathbf{s},$$

ϵ_1 and ϵ_2 being the extension ratios for the directions of $\mathbf{s}$ and $\mathbf{t}$ respectively.

96. Extension ellipses. Consider the elementary circle with centre P and radius η, lying in the tangent plane at P. For points Q on the circumference of this circle we have

$$(d\mathbf{r})^2 = \eta^2 \quad \text{.........................(6)},$$

which, in virtue of (5), may be written

$$d\mathbf{r}' \cdot (\Phi_c^{-1} \cdot \Phi^{-1}) \cdot d\mathbf{r}' = \eta^2 \quad \text{.................(7)}.$$

Now, since P' is the origin of the vector $d\mathbf{r}'$, this equation shows that Q' lies on an ellipse with centre at P', lying in the tangent plane at that point. Thus points lying on the small circle (6) with

* If Φ is planar (but not linear) in the tangent plane at P, it may be made complete by the addition of the dyad $\mathbf{n}\,\mathbf{n}$. This dyad contributes nothing to the value of the second member of (2), since $\mathbf{n}$ is perpendicular to $d\mathbf{r}$.

centre at P are represented by points on the small ellipse (7) with centre at P'. We may call the latter the *extension ellipse* for the point P. From (5′) and (7) it then follows that:

The extension ratio for any direction at P varies as the radius of the extension ellipse in the corresponding direction at P'.

Similarly it is evident from (2) that points of S, which lie on the elementary ellipse

$$d\mathbf{r} \cdot (\Phi_c \cdot \Phi) \cdot d\mathbf{r} = \eta^2 \qquad \text{.....................(8)},$$

with centre at P, correspond to points of S' lying on the small circle

$$(d\mathbf{r}')^2 = \eta^2,$$

with centre at P'. The ellipse (8) may be called the *reciprocal extension ellipse* for the representation. From (4) and (8) it is then clear that:

The extension ratio for any direction at P varies inversely as the radius of the ellipse (8) *in that direction.*

97. Principal directions for the representation. Let the surface directions $\mathbf{s}, \mathbf{t}$ at P correspond to the directions $\mathbf{s}', \mathbf{t}'$ at P'. The latter will be perpendicular if $\mathbf{s}' \cdot \mathbf{t}' = 0$, that is to say provided

$$\mathbf{s} \cdot (\Phi_c \cdot \Phi) \cdot \mathbf{t} = 0.$$

This requires that $\mathbf{s}$ and $\mathbf{t}$ be parallel to conjugate diameters of the reciprocal extension ellipse. Thus:

Directions on S, which are parallel to conjugate diameters of the ellipse (8), *are represented by perpendicular directions on S'.*

Now the principal axes of an ellipse are the only pair of perpendicular conjugate diameters. Thus there is one and only one pair of perpendicular surface directions at P which are represented by perpendicular directions on S'. These are called the *principal directions* at P for the representation. The *principal lines* for the transformation are the orthogonal system whose directions at any point are the principal directions for the transformation. And, since the principal axes of the ellipse (8) are its greatest and least diameters, it follows that:

The principal directions at any point for the representation are those of least and greatest extension.

The extension ratios in these directions may be called the principal ratios, and will be denoted by ϵ_1 and ϵ_2. Then, since the extension ratio in any direction varies inversely as the radius of the ellipse (8) in that direction, it follows that the ratio for the direction inclined at an angle θ to the principal direction whose ratio is ϵ_1 is given by

$$\epsilon^2 = \epsilon_1^2 \cos^2\theta + \epsilon_2^2 \sin^2\theta,$$

and also that:

The sum of the squares of the extension ratios for two perpendicular directions at a point is invariant and equal to $\epsilon_1^2 + \epsilon_2^2$.

The value of this invariant is expressible in terms of Φ. Let Ψ be the dyadic

$$\Psi = \Phi \cdot \nabla \mathbf{r} \quad\text{.........................(9)},$$

obtainable from Φ by omitting the normal components of the consequents, and retaining only the components tangential to S. Then the relation (2) may be equally well expressed

$$d\mathbf{r}' = \Psi \cdot d\mathbf{r},$$

since the dyads omitted contribute nothing to the value of $\Phi \cdot d\mathbf{r}$. Similarly (4) may be written

$$\epsilon^2 = \mathbf{t} \cdot (\Psi_c \cdot \Psi) \cdot \mathbf{t} \quad\text{....................(10)}.$$

The dyadic $\Psi_c \cdot \Psi$ has all its antecedents and consequents parallel to the tangent plane at P; and if $\mathbf{s}$, $\mathbf{t}$ are perpendicular unit surface vectors, the scalar of the dyadic is given by

$$(\Psi_c \cdot \Psi)_s = \mathbf{s} \cdot (\Psi_c \cdot \Psi) \cdot \mathbf{s} + \mathbf{t} \cdot (\Psi_c \cdot \Psi) \cdot \mathbf{t},$$

and is therefore equal to the sum of the squares of the extension ratios for two perpendicular directions. The value of the above invariant is therefore given by

$$\epsilon_1^2 + \epsilon_2^2 = (\Psi_c \cdot \Psi)_s \quad\text{....................(11)}.$$

Expressed in terms of the principal extension ratios and the unit vectors $\mathbf{a}$, $\mathbf{b}$ in the principal directions of representation, the dyadic $\Psi_c \cdot \Psi$ becomes

$$\Psi_c \cdot \Psi = \epsilon_1^2 \mathbf{a}\,\mathbf{a} + \epsilon_2^2 \mathbf{b}\,\mathbf{b} \quad\text{.................(12)}.$$

The directions at P' corresponding to those of $\mathbf{a}$ and $\mathbf{b}$ are given by $\Psi \cdot \mathbf{a}$ and $\Psi \cdot \mathbf{b} = \Psi \cdot (\mathbf{n} \times \mathbf{a})$. Since these are perpendicular we have

$$\mathbf{a} \cdot (\Psi_c \cdot \Psi \times \mathbf{n}) \cdot \mathbf{a} = 0,$$

and similarly $$\mathbf{b} \cdot (\Psi_c \cdot \Psi \times \mathbf{n}) \cdot \mathbf{b} = 0.$$

Thus *the principal directions for the representation are those of the asymptotes of the conic*

$$\mathbf{R}\cdot(\Psi_c\cdot\Psi\times\mathbf{n})\cdot\mathbf{R}=1 \quad\text{.................(13).}$$

Again *the ratio of corresponding elements of area* of S' and S is the ratio of the area of the circle of radius η to that of the ellipse (8). This ratio, μ, is the product of the principal extension ratios. Now the second of the dyadic $\Psi_c\cdot\Psi$, in virtue of (12), is given by

$$(\Psi_c\cdot\Psi)_2=\epsilon_1^2\epsilon_2^2\,\mathbf{n}\,\mathbf{n},$$

and the scalar of this has the value

$$(\Psi_c\cdot\Psi)_{2s}=\epsilon_1^2\epsilon_2^2 \quad\text{....................(14),}$$

which is the square of the ratio μ. When this ratio is constant the representation is characterised by proportionality of areas; and, when this constant is equal to unity, the representation is said to be *equivalent.*

If each of the principal extension ratios is constant, the representation is said to be *uniform*. In this case the ellipse (8) has the same shape for every point P, and each of the invariants $(\Psi_c\cdot\Psi)_s$ and $(\Psi_c\cdot\Psi)_{2s}$ is constant. *Conformal representation* is characterised by the property that the extension ratio has the same value for all directions at a given point, but may vary from point to point.

98. Fundamental magnitudes and the operator ∇' on S'. Let corresponding points on the two surfaces be represented by the same values of the coordinates u, v. Then, if suffixes 1, 2 denote partial differentiations with respect to u, v, it follows from (2) that

$$\mathbf{r}_1'=\Phi\cdot\mathbf{r}_1,\quad \mathbf{r}_2'=\Phi\cdot\mathbf{r}_2 \quad\text{.................(15).}$$

The first order magnitudes E', F', G' for the surface S' are given by

$$\left.\begin{aligned}E'&=\mathbf{r}_1'^2 &&=\mathbf{r}_1\cdot\Phi_c\cdot\Phi\cdot\mathbf{r}_1\\ F'&=\mathbf{r}_1'\cdot\mathbf{r}_2' &&=\mathbf{r}_1\cdot\Phi_c\cdot\Phi\cdot\mathbf{r}_2\\ G'&=\mathbf{r}_2'^2 &&=\mathbf{r}_2\cdot\Phi_c\cdot\Phi\cdot\mathbf{r}_2\end{aligned}\right\} \quad\text{..............(16),}$$

whence H'^2 may be calculated. The unit normal $\mathbf{n}'$ to the surface S' is found from the formula

$$H'\mathbf{n}'=\mathbf{r}_1'\times\mathbf{r}_2'=\mathbf{r}_1\cdot\Phi_c\times\Phi\cdot\mathbf{r}_2 \quad\text{...........(17),}$$

so that

$$H'^2=(\mathbf{r}_1\cdot\Phi_c\times\Phi\cdot\mathbf{r}_2)^2 \quad\text{.................(18).}$$

We may express the differential operator ∇', for the surface S', in terms of ∇ and the dyadic Φ. Let the principal lines for the representation be taken as parametric curves. Since these are orthogonal on both surfaces we have

$$\nabla' = \frac{1}{E'}\mathbf{r}_1'\frac{\partial}{\partial u} + \frac{1}{G'}\mathbf{r}_2'\frac{\partial}{\partial v},$$

and therefore, in virtue of (15),

$$\nabla' = \Phi \cdot \left(\frac{1}{E'}\mathbf{r}_1\mathbf{r}_1 + \frac{1}{G'}\mathbf{r}_2\mathbf{r}_2\right)\cdot\left(\frac{1}{E}\mathbf{r}_1\frac{\partial}{\partial u} + \frac{1}{G}\mathbf{r}_2\frac{\partial}{\partial v}\right)$$

$$= \Phi \cdot \left(\frac{E}{E'}\mathbf{a}\,\mathbf{a} + \frac{G}{G'}\mathbf{b}\,\mathbf{b}\right)\cdot \nabla,$$

a and **b** being the unit tangents to the parametric curves. Now E'/E and G'/G are the squares of the principal extension ratios ϵ_1 and ϵ_2. Consequently, in virtue of (12), the last equation may be expressed

$$\nabla' = \Phi \cdot (\Psi_c \cdot \Psi)^{-1} \cdot \nabla \quad \ldots\ldots\ldots\ldots\ldots\ldots(19),$$

where $(\Psi_c \cdot \Psi)^{-1}$ is the uniplanar reciprocal (Art. 78) of the planar dyadic (12). Formula (19) gives the required expression for ∇'.

In the case of *conformal representation* $\epsilon_1 = \epsilon_2 = \epsilon$, a point-function for the surface. The condition that the representation may be conformal is therefore

$$\Psi_c \cdot \Psi = \epsilon^2 \nabla \mathbf{r}.$$

When this relation holds it follows that

$$\nabla' = \epsilon^{-2}\Phi \cdot \nabla.$$

In order that two surfaces may be *applicable*, there must be a conformal representation with ϵ identically equal to unity. This requires

$$\Psi_c \cdot \Psi = \nabla \mathbf{r}.$$

In this case
$$\nabla' = \Phi \cdot \nabla.$$

Lastly, suppose that the surface S' has been derived from S by the linear transformation of Art. 95. For any assigned family of curves on S, the geometrical properties of the corresponding family on S' may be deduced from the above formulae. If **t** is the unit tangent to the given family on S, that to the corresponding family on S' is

$$\mathbf{t}' = \epsilon^{-1}\Phi \cdot \mathbf{t}.$$

The operator ∇' for S' is given by (19); and the properties of the family on S' are then found from the formulae of Chapter II. Similarly, if the curves are specified by an equation of the form $\phi = \text{const.}$, the formulae of Chapter III will give the required results.

Conical Projection

99. Conical projection in general. As an illustration of the above theory, suppose that the surface S' is derived from S by the transformation

$$\mathbf{r}' = \phi\mathbf{r} \quad \text{.........................(20)},$$

where ϕ is a point-function on S. Then the point P' is the conical

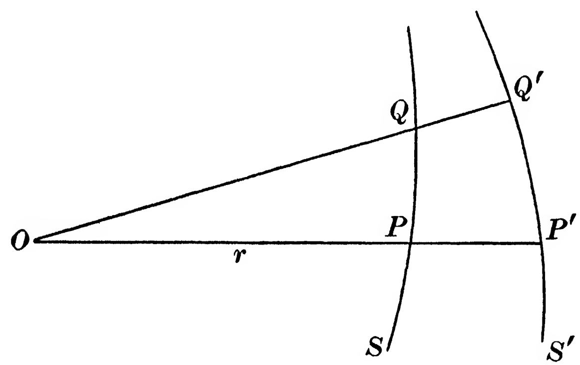

Fig. 9.

projection of P on S', with the origin O as vertex of projection. From (20) it follows that

$$\begin{aligned} d\mathbf{r}' &= d\phi\,\mathbf{r} + \phi\,d\mathbf{r} = d\mathbf{r}\cdot\nabla\phi\,\mathbf{r} + \phi\,d\mathbf{r} \\ &= (\mathbf{r}\nabla\phi + \phi\mathrm{I})\cdot d\mathbf{r}, \end{aligned}$$

where I is the unit dyadic, or idemfactor, of Art. 69. Hence, in this case,

$$\Phi = \phi\mathrm{I} + \mathbf{r}\nabla\phi \quad \text{......................(21)},$$

and
$$\Phi_c\cdot\Phi = \phi^2\mathrm{I} + \phi(\mathbf{r}\nabla\phi + \nabla\phi\,\mathbf{r}) + r^2\nabla\phi\nabla\phi \quad \text{......(22)},$$

where r is the length OP, so that $r^2 = \mathbf{r}\cdot\mathbf{r}$. In virtue of (4) the extension ratio for the direction of the unit surface vector $\mathbf{t}$ at P is given by

$$\epsilon^2 = \phi^2 + 2\phi(\mathbf{r}\cdot\mathbf{t})(\mathbf{t}\cdot\nabla\phi) + r^2(\mathbf{t}\cdot\nabla\phi)^2 \quad \text{.........(23)},$$

and the direction on S' corresponding to $\mathbf{t}$ on S is that of the unit vector

$$\mathbf{t}' = \epsilon^{-1}[\phi\mathbf{t} + (\mathbf{t}\cdot\nabla\phi)\,\mathbf{r}] \qquad (24).$$

The planar dyadic Ψ defined by (9) has the value

$$\Psi = \phi\nabla\mathbf{r} + \mathbf{r}\nabla\phi \qquad (25),$$

so that

$$\Psi_c\cdot\Psi = \phi^2\nabla\mathbf{r} + \phi(\mathbf{r}\cdot\nabla\mathbf{r}\nabla\phi + \nabla\phi\mathbf{r}\cdot\nabla\mathbf{r}) + r^2\nabla\phi\nabla\phi.$$

And, since the sum of the squares of the extension ratios for two perpendicular directions is equal to the scalar of this dyadic, we have

$$\epsilon_1^2 + \epsilon_2^2 = 2\phi^2 + 2\mathbf{r}\cdot\nabla\phi + r^2(\nabla\phi)^2 \qquad (26).$$

With our usual notation for partial derivatives we have

$$\mathbf{r}_1' = \phi_1\mathbf{r} + \phi\mathbf{r}_1, \quad \mathbf{r}_2' = \phi_2\mathbf{r} + \phi\mathbf{r}_2,$$

so that the first order magnitudes on S' have the values

$$\left.\begin{aligned} E' &= \phi_1^2 r^2 + 2\phi\phi_1 r r_1 + E\phi^2 \\ F' &= \phi_1\phi_2 r^2 + r\phi(r_1\phi_2 + r_2\phi_1) + F\phi^2 \\ G' &= \phi_2^2 r^2 + 2\phi\phi_2 r r_2 + G\phi^2 \end{aligned}\right\} \qquad (27).$$

The unit normal $\mathbf{n}'$ to S' is given by

$$\begin{aligned} H'\mathbf{n}' &= (\phi_1\mathbf{r} + \phi\,\mathbf{r}_1)\times(\phi_2\mathbf{r} + \phi\,\mathbf{r}_2) \\ &= \phi^2 H\mathbf{n} + \phi\mathbf{r}\times(\phi_1\mathbf{r}_2 - \phi_2\mathbf{r}_1). \end{aligned}$$

Now the expression in brackets* is equal to $H\mathbf{n}\times\nabla\phi$. Also H'/H is the ratio μ of the areas of corresponding elements of S' and S. Hence the above equation may be written

$$\mu\mathbf{n}' = \phi(\phi + \mathbf{r}\cdot\nabla\phi)\,\mathbf{n} - \phi p\nabla\phi \qquad (28),$$

where $p = \mathbf{r}\cdot\mathbf{n}$, being the perpendicular distance from O to the tangent plane at P. By "squaring" both numbers of (28) we find for μ^2 the value

$$\mu^2 = \phi^2[(\phi + \mathbf{r}\cdot\nabla\phi)^2 + p^2(\nabla\phi)^2] \qquad (29).$$

100. A particular case. Suppose that ϕ is a function of r only, as in the case of Inversion (Vol. I, Art. 82). Then

$$\phi_1 = \phi' r_1, \quad \nabla\phi = \phi'\nabla r,$$

* Examples I, 16.

and so on, ϕ' denoting the derivative of ϕ with respect to r. The equation (28) becomes

$$\mu\mathbf{n}' = \phi(\phi + \phi'\,\mathbf{r}\cdot\nabla r)\,\mathbf{n} - \phi\phi' p\nabla r \quad \text{.........(30)},$$

from which it follows that

$$\mu^2 = \phi^2[\phi^2 + 2\phi\phi'\,\mathbf{r}\cdot\nabla r + \phi'^2 r^2(\nabla r)^2] \quad \text{.........(31)}.$$

Now the three-parametric gradient of r in space is a unit vector in the direction of $\mathbf{r}$. The gradient of r on the surface S is the component of this unit vector tangential to the surface. Its magnitude is therefore $\sin\theta$, where θ is the inclination of $\mathbf{r}$ to $\mathbf{n}$. Consequently $\mathbf{r}\cdot\nabla r$ is equal to $r\sin^2\theta$, and we may write the last equation as

$$\mu^2 = \phi^2[\phi^2 + r\phi'\sin^2\theta\,(2\phi + r\phi')] \quad \text{...........(32)}.$$

Similarly (30) may be expressed in the alternative form

$$\mu\mathbf{n}' = \phi(\phi + r\phi')\,\mathbf{n} - \frac{p}{r}\phi\phi'\mathbf{r} \quad \text{..............(33)}.$$

Moreover, since $\mathbf{r}^2 = r^2$, it follows that

$$\mathbf{r}\cdot\nabla\mathbf{r} = r\nabla r.$$

Consequently in this case we have

$$\Psi_c\cdot\Psi = \phi^2\nabla\mathbf{r} + r\phi'(2\phi + r\phi')(\nabla r\nabla r),$$

and the sum of the squares of the extension ratios for two perpendicular directions on S is given by

$$(\Psi_c\cdot\Psi)_s = 2\phi^2 + r\phi'(2\phi + r\phi')(\nabla r)^2 \quad \text{.........(34)}.$$

Again,

$$\mathbf{r}_1' = \phi' r_1\mathbf{r} + \phi\mathbf{r}_1,$$
$$\mathbf{r}_2' = \phi' r_2\mathbf{r} + \phi\mathbf{r}_2,$$

and therefore, since $\mathbf{r}\cdot\mathbf{r}_1 = rr_1$ and $\mathbf{r}\cdot\mathbf{r}_2 = rr_2$, the first order magnitudes for S' have the values

$$\left.\begin{aligned} E' &= \phi' r_1^2(\phi' r^2 + 2\phi r) + E\phi^2 \\ F' &= \phi' r_1 r_2(\phi' r^2 + 2\phi r) + F\phi^2 \\ G' &= \phi' r_2^2(\phi' r^2 + 2\phi r) + G\phi^2 \end{aligned}\right\} \quad \text{...........(35)}.$$

If the parametric curves are orthogonal on S $(F=0)$, they will project into orthogonal curves on S' provided

$$\phi' r_1 r_2(\phi' r^2 + 2\phi r) = 0 \quad \text{.................(36)}.$$

This condition is satisfied in the following cases: (i) when ϕ is constant, the two surfaces being then similar and similarly situated,

with O as centre of similarity; (ii) when $\phi' r^2 + 2\phi r = 0$, leading to $\phi = c^2/r^2$, which is the case of inversion already considered; (iii) when either r_1 or r_2 vanishes identically, that is to say, when one family of the orthogonal system on S consists of spherical curves, each lying on a sphere with centre at the origin O.

Lastly, let us find the second order magnitude M' on S'. On differentiating the relation $\mathbf{r} \cdot \mathbf{r}_1 = rr_1$ with respect to v we have

$$F + \mathbf{r} \cdot \mathbf{r}_{12} = rr_{12} + r_1 r_2 \quad \text{.................. (37).}$$

Then, in virtue of (33), the magnitude M' is given by

$$\begin{aligned}\mu M' &= \mu \mathbf{n}' \cdot \mathbf{r}'_{12} \\ &= \left[\phi(\phi + r\phi')\,\mathbf{n} - \frac{p}{r}\phi\phi'\mathbf{r}\right] \\ &\qquad \cdot \left[(\phi'' r_1 r_2 + \phi' r_{12})\,\mathbf{r} + \phi \mathbf{r}_{12} + \phi'(r_1 \mathbf{r}_2 + r_2 \mathbf{r}_1)\right],\end{aligned}$$

which, by means of (37), may be reduced to the form

$$\mu M' = \phi^2(\phi + r\phi')\,M + \phi p\left[\phi\phi'' r_1 r_2 - \frac{1}{r}\phi'(2\phi' r r_1 r_2 - \phi F + \phi r_1 r_2)\right].$$

If the parametric curves on S are lines of curvature, $F = M = 0$. These will project into lines of curvature on S' provided $F' = M' = 0$, that is to say, provided (36) holds simultaneously with

$$r_1 r_2 (r\phi\phi'' - 2\phi'^2 r - \phi\phi') = 0.$$

This equation is also satisfied by the solutions found above for (36). Consequently:

When ϕ is a function of r only, the lines of curvature on S project into lines of curvature on S' (a) in the case of inversion, with O as centre of inversion ($\phi = c^2/r^2$), (b) when the surfaces are similar and similarly situated, with O as centre of similarity ($\phi =$ const.), and (c) when either family of lines of curvature on S is a family of spherical curves, lying on spheres with a common centre at O.

EXAMPLES X

1. If ϵ_1 and ϵ_2 are the extension ratios for any two perpendicular directions, parallel to the unit vectors $\mathbf{a}$ and $\mathbf{b}$, the ratio ϵ for the direction inclined at an angle θ to $\mathbf{a}$ is given by

$$\epsilon^2 = (\mathbf{a}\cos\theta + \mathbf{b}\sin\theta)\cdot(\Phi_c\cdot\Phi)\cdot(\mathbf{a}\cos\theta+\mathbf{b}\sin\theta)$$
$$= \epsilon_1{}^2\cos^2\theta + \epsilon_2{}^2\sin^2\theta + \epsilon_1\epsilon_2\cos\phi'\sin 2\theta,$$

where ϕ' is the angle between the directions on S' which correspond to those of $\mathbf{a}$ and $\mathbf{b}$.

2. Show that, if ϑ is the inclination of the normals at corresponding points in the general case of conical projection,

$$\mu\cos\vartheta = \phi^2 + \tfrac{1}{2}\mathbf{r}\cdot\nabla\phi^2.$$

3. Prove that, for conical projection in general, the fundamental magnitudes of the second order for S' are given by

$$\mu L' = (\phi + \mathbf{r}\cdot\nabla\phi)\,\phi^2 L - \phi p\,(2\phi_1\mathbf{r}_1 + \phi\mathbf{r}_{11})\cdot\nabla\phi + \phi^2\phi_{11}\,p,$$
$$\mu M' = (\phi + \mathbf{r}\cdot\nabla\phi)\,\phi^2 M - \phi p\,(\phi_1\mathbf{r}_2 + \phi_2\mathbf{r}_1 + \phi\mathbf{r}_{12})\cdot\nabla\phi + \phi^2\phi_{12}\,p,$$

and a similar formula for N'.

4. For the particular case of conical projection in which $\phi = \phi(r)$, show that

$$\mu\mathbf{n}' = \phi^2\mathbf{n} - \frac{1}{r}\,\phi\phi'\mathbf{r}\times(\mathbf{r}\times\mathbf{n}),$$

and also that the second order magnitude L' is given by

$$\mu L' = \phi^2(\phi + r\phi')\,L + \phi p\left[\phi\phi'' r_1{}^2 - \frac{1}{r}\,\phi'\,(2\phi' r r_1{}^2 - \phi E + \phi r_1{}^2)\right].$$

Write down the corresponding formula for N'.

CHAPTER XI

SMALL DEFORMATIONS OF CURVES AND SURFACES

SMALL DEFORMATIONS OF CURVES*

101. Single twisted curve. We proceed to consider small deformations of curves and surfaces, in which the displacement $\mathbf{s}$ of each point is a small quantity of the first order, and quantities of higher order are negligible †.

Consider first a given curve in space. The position vector $\mathbf{r}$ of a point on the curve may be regarded as a function of the arc-length s of the curve, measured from a fixed point on it. Let $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ be the unit tangent, principal normal and binormal. These are connected with the curvature κ and the torsion τ according to the Serret-Frenet formulae. Imagine a small deformation of the curve, such that the point of the curve originally at $\mathbf{r}$ suffers a small displacement $\mathbf{s}$, its new position vector $\mathbf{r}_1$ being then

$$\mathbf{r}_1 = \mathbf{r} + \mathbf{s} \qquad \text{.........................(1).}$$

Let a suffix unity be used to distinguish quantities belonging to the deformed curve, and let accents denote differentiations with respect to the arc-length s. Then the element $d\mathbf{r}_1$ of the deformed curve, corresponding to the element $d\mathbf{r}$ of the original, is given by $d\mathbf{r}_1 = d\mathbf{r} + d\mathbf{s}$, and its length ds_1 by

$$(ds_1)^2 = (d\mathbf{r}_1)^2 = (d\mathbf{r})^2 + 2d\mathbf{r} \cdot d\mathbf{s} = ds^2 (1 + 2\mathbf{t} \cdot \mathbf{s}').$$

Consequently $\qquad ds_1 = ds\,(1 + \mathbf{t} \cdot \mathbf{s}').$

The quantity $\mathbf{t} \cdot \mathbf{s}'$ represents the increase of length per unit length of the curve, or the *modulus of extension* of the curve at the point considered. Let it be denoted by ϵ. Then

$$ds_1 = ds\,(1 + \epsilon), \quad ds = ds_1 (1 - \epsilon) \qquad \text{...........(2).}$$

* Arts. 101–102 are taken from a paper by the author "On Small Deformations of Curves," *Bulletin of the Amer. Math. Society*, Jan.–Feb. 1927, pp. 58–62.

† See also a paper by Perna, *Giornale di Matematiche*, Vol. 36 (1898), pp. 286–299; and another by Salkowski, *Mathematische Annalen*, Vol. 66 (1908), pp. 517–557.

ınit tangent $\mathbf{t}_1$ to the deformed curve is given by

$$\mathbf{t}_1 = \frac{d\mathbf{r}_1}{ds_1} = (1-\epsilon)(\mathbf{r}' + \mathbf{s}') = (1-\epsilon)\mathbf{t} + \mathbf{s}' \quad \text{.........(3).}$$

ısequently, if κ_1 is the curvature and $\mathbf{n}_1$ the unit principal ormal for the new curve,

$$\kappa_1 \mathbf{n}_1 = \frac{d\mathbf{t}_1}{ds_1} = (1-\epsilon)\frac{d\mathbf{t}_1}{ds} = (1-2\epsilon)\kappa\mathbf{n} - \epsilon'\mathbf{t} + \mathbf{s}''.$$

On "squaring" both members, and neglecting small quantities of the second order, we have

$$\kappa_1^2 = (1-4\epsilon)\kappa^2 + 2\kappa\mathbf{n}\cdot\mathbf{s}'',$$

and therefore

$$\kappa_1 = (1-2\epsilon)\kappa + \mathbf{n}\cdot\mathbf{s}'' \quad \text{....................(4).}$$

Inserting this value in the above product $\kappa_1\mathbf{n}_1$ we find

$$\left.\begin{aligned} \mathbf{n}_1 &= \mathbf{n} - \frac{1}{\kappa}[(\mathbf{n}\cdot\mathbf{s}'')\mathbf{n} + \epsilon'\mathbf{t} - \mathbf{s}''] \\ &= \mathbf{n} - (\mathbf{n}\cdot\mathbf{s}')\mathbf{t} + \frac{1}{\kappa}(\mathbf{b}\cdot\mathbf{s}'')\mathbf{b} \end{aligned}\right\} \quad \text{............(5).}$$

The unit binormal $\mathbf{b}$ to the deformed curve is then given by

$$\left.\begin{aligned} \mathbf{b}_1 = \mathbf{t}_1 \times \mathbf{n}_1 &= (1-\epsilon)\mathbf{b} + \mathbf{s}' \times \mathbf{n} - \frac{1}{\kappa}(\mathbf{b}\cdot\mathbf{s}'')\mathbf{n} \\ &= \mathbf{b} - (\mathbf{b}\cdot\mathbf{s}')\mathbf{t} - \frac{1}{\kappa}(\mathbf{b}\cdot\mathbf{s}'')\mathbf{n} \end{aligned}\right\} \quad \text{......(6).}$$

The torsion τ_1 may then be found by differentiating the unit binormal, using the Serret-Frenet formulae. Thus on differentiating (6), and using the preceding results, we find on reduction

$$\tau_1 = (1-\epsilon)\tau + \kappa\mathbf{b}\cdot\mathbf{s}' + \frac{d}{ds}\left(\frac{1}{\kappa}\mathbf{b}\cdot\mathbf{s}''\right) \quad \text{.........(7).}$$

We have thus determined the geometric characteristics of the deformed curve in terms of those of the original curve and the small displacement $\mathbf{s}$. The rectangular trihedral, consisting of the unit tangent, the principal normal, and the binormal, undergoes a small rotation which is represented by the vector

$$\mathbf{R} = \frac{1}{\kappa}(\mathbf{b}\cdot\mathbf{s}'')\mathbf{t} - (\mathbf{b}\cdot\mathbf{s}')\mathbf{n} + (\mathbf{n}\cdot\mathbf{s}')\mathbf{b} \quad \text{........(8),}$$

or

$$\mathbf{R} = \frac{1}{\kappa}(\mathbf{b}\cdot\mathbf{s}'')\mathbf{t} + \mathbf{t}\times\mathbf{s}' \quad \text{.........................(9).}$$

The coefficients of $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ in (8) represent the small rotations of the trihedral about the tangent, the principal normal, and the binormal, respectively.

An *inextensional deformation* of the curve is one for which ϵ vanishes identically*. If $\mathbf{s}$ is expressed in the form

$$\mathbf{s} = P\mathbf{t} + Q\mathbf{n} + R\mathbf{b},$$

the vanishing of $\mathbf{t} \cdot \mathbf{s}'$ gives $P' = \kappa Q$ as the necessary and sufficient condition for inextensional deformation.

102. Family of curves on a surface. Consider next a family of curves on a given surface. Suppose that the curves of the family suffer a small deformation, but remain on the same surface. Then the small displacement $\mathbf{s}$ at any point is tangential to the surface, and is a function of two parameters that specify the point considered. On any one curve the vector $\mathbf{s}$ is a function of the arc-length s, and the formulae found above hold for the deformed curve.

Other results may be very neatly expressed in terms of two-parametric differential invariants for the surface. If ϕ is the value of any function associated with the deformed curve at the point $\mathbf{r} + \mathbf{s}$, the new value of this function at the point $\mathbf{r}$ originally occupied by this point of the curve is $\phi - \mathbf{s} \cdot \nabla\phi$. Thus, after the deformation, the unit tangent $\bar{\mathbf{t}}$ to the new curve through the point $\mathbf{r}$ is, in virtue of (3),

$$\left.\begin{aligned}\bar{\mathbf{t}} &= \mathbf{t}_1 - \mathbf{s} \cdot \nabla\mathbf{t}_1 = (1-\epsilon)\,\mathbf{t} + \mathbf{t} \cdot \nabla\mathbf{s} - \mathbf{s} \cdot \nabla\mathbf{t} \\ &= (1-\epsilon)\,\mathbf{t} + \operatorname{rot}(\mathbf{s} \times \mathbf{t}) - \mathbf{s}\operatorname{div}\mathbf{t} + \mathbf{t}\operatorname{div}\mathbf{s}\end{aligned}\right\} \quad \ldots\ldots(10).$$

The *line of striction* of the deformed family is given by $\operatorname{div}\bar{\mathbf{t}} = 0$, which may be expressed in the form

$$(1-\epsilon)\operatorname{div}\mathbf{t} - \mathbf{t} \cdot \nabla\epsilon - \mathbf{s} \cdot \nabla \operatorname{div}\mathbf{t} + \mathbf{t} \cdot \nabla \operatorname{div}\mathbf{s} = 0 \quad \ldots(11),$$

since the divergence of the rotation of the normal vector $\mathbf{s} \times \mathbf{t}$ vanishes identically (Art. 4). If the original family of curves is one of parallels, $\operatorname{div}\mathbf{t}$ vanishes identically. Hence *a family of parallels will remain parallels after the deformation provided*

$$\mathbf{t} \cdot \nabla(\epsilon - \operatorname{div}\mathbf{s}) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(12).$$

* This case is considered in some detail by Sannia, *Rendiconti di Palermo*, Vol. 21 (1906), pp. 229–256.

The geodesic curvature of a curve of the original family is $\mathbf{n}\cdot\operatorname{rot}\mathbf{t}$, and that of one of the deformed curves is

$$\mathbf{n}\cdot\operatorname{rot}[(1-\epsilon)\mathbf{t}+\mathbf{t}\cdot\nabla\mathbf{s}-\mathbf{s}\cdot\nabla\mathbf{t}].$$

If then the family of curves is a family of geodesics, it will remain so after the deformation provided

$$\mathbf{n}\cdot[\mathbf{t}\times\nabla\epsilon+\operatorname{rot}(\mathbf{t}\cdot\nabla\mathbf{s}-\mathbf{s}\cdot\nabla\mathbf{t})]=0.$$

Similarly, the original family will be lines of curvature if

$$\mathbf{t}\cdot\operatorname{rot}\mathbf{t}=0;$$

and they will remain lines of curvature after the deformation provided

$$\bar{\mathbf{t}}\cdot\operatorname{rot}\bar{\mathbf{t}}=0.$$

SMALL DEFORMATIONS OF SURFACES*

103. First order magnitudes. Modulus of dilation. Consider next a small deformation of a given surface S. Suppose that the point, whose original position vector is $\mathbf{r}$, undergoes a small displacement $\mathbf{s}$, which is a point-function for the surface. As before, $\mathbf{s}$ will be regarded as a small quantity of the first order, quantities of higher order being negligible. The position vector $\mathbf{r}'$ of the corresponding point on the deformed surface S' is given by

$$\mathbf{r}'=\mathbf{r}+\mathbf{s} \quad\quad \text{.........................(1).}$$

Let corresponding points of the surfaces S, S' be characterised by the same values of the parameters, and let suffixes 1, 2 denote differentiations with respect to these parameters. Then, if E, F, G are the first order magnitudes for S, those for S' have the values

$$\left.\begin{aligned} E'&=\mathbf{r}_1'^2=E+2\mathbf{r}_1\cdot\mathbf{s}_1\\ F'&=\mathbf{r}_1'\cdot\mathbf{r}_2'=F+\mathbf{r}_1\cdot\mathbf{s}_2+\mathbf{r}_2\cdot\mathbf{s}_1\\ G'&=\mathbf{r}_2'^2=G+2\mathbf{r}_2\cdot\mathbf{s}_2\end{aligned}\right\} \quad\quad \text{............(2).}$$

Consequently, since small quantities of the second and higher orders may be neglected,

$$\begin{aligned} H'^2&=E'G'-F'^2\\ &=EG-F^2+2[E\mathbf{r}_2\cdot\mathbf{s}_2+G\mathbf{r}_1\cdot\mathbf{s}_1-F(\mathbf{r}_1\cdot\mathbf{s}_2+\mathbf{r}_2\cdot\mathbf{s}_1)]\\ &=H^2+2H^2\operatorname{div}\mathbf{s}.\end{aligned}$$

* The substance of Arts. 103–111 was given by the author in two papers "On small Deformation of Surfaces, etc.," *Quarterly Journal*, Vol. 50 (1925), pp. 272–296, and *Messenger of Mathematics*, Vol. 57 (1927), pp. 106–112.

Hence, to the first order,

$$H' = H(1 + \operatorname{div} \mathbf{s}) \quad \text{.....................(3).}$$

Now the areas of corresponding elements of S and S' are $H\,du\,dv$ and $H'\,du\,dv$. Hence the increase of area per unit area of S is equal to*

$$\frac{H' - H}{H} = \operatorname{div} \mathbf{s}.$$

This quantity may be called the *modulus of dilation* of S due to the deformation, or briefly the *dilation*. It will be convenient to denote it by θ. Thus

$$\theta = \operatorname{div} \mathbf{s} \quad \text{..........................(4).}$$

104. Unit normal to the deformed surface. The unit normal $\mathbf{n}'$ to the surface S' is given by

$$\begin{aligned}
\mathbf{n}' &= \frac{1}{H'}(\mathbf{r}_1 + \mathbf{s}_1) \times (\mathbf{r}_2 + \mathbf{s}_2) \\
&= \frac{1}{H(1+\theta)}[\mathbf{r}_1 \times \mathbf{r}_2 + (\mathbf{r}_1 \times \mathbf{s}_2 - \mathbf{r}_2 \times \mathbf{s}_1)].
\end{aligned}$$

We lose no generality by choosing orthogonal parametric curves, for the final result will be expressed in invariant form. Then $H = \sqrt{(EG)}$, and we may write

$$\begin{aligned}
\frac{1}{H}(\mathbf{r}_1 \times \mathbf{s}_2 - \mathbf{r}_2 \times \mathbf{s}_1) &= \frac{1}{G}(\mathbf{r}_2 \times \mathbf{n}) \times \mathbf{s}_2 - \frac{1}{E}(\mathbf{n} \times \mathbf{r}_1) \times \mathbf{s}_1 \\
&= \frac{1}{G}[(\mathbf{r}_2 \cdot \mathbf{s}_2)\,\mathbf{n} - (\mathbf{n} \cdot \mathbf{s}_2)\,\mathbf{r}_2] - \frac{1}{E}[(\mathbf{n} \cdot \mathbf{s}_1)\,\mathbf{r}_1 - (\mathbf{r}_1 \cdot \mathbf{s}_1)\,\mathbf{n}] \\
&= \mathbf{n}\left(\frac{1}{E}\mathbf{r}_1 \cdot \mathbf{s}_1 + \frac{1}{G}\mathbf{r}_2 \cdot \mathbf{s}_2\right) - \mathbf{n} \times \left(\frac{1}{E}\mathbf{r}_1 \times \mathbf{s}_1 + \frac{1}{G}\mathbf{r}_2 \times \mathbf{s}_2\right) \\
&= \mathbf{n} \operatorname{div} \mathbf{s} - \mathbf{n} \times \operatorname{rot} \mathbf{s}.
\end{aligned}$$

Substituting this value in the above formula for $\mathbf{n}'$, and neglecting small quantities of higher order than the first, we have the required expression

$$\mathbf{n}' = \mathbf{n} - \mathbf{n} \times \operatorname{rot} \mathbf{s} \quad \text{.....................(5).}$$

Thus the change in the unit normal is the same as if the element of surface were rotated by an amount $\operatorname{rot} \mathbf{s}$. We shall see later,

* It will be shown in Art. 108 that many of these formulae may be deduced from the theory of Representation of Surfaces as given in Chapter X. However, the more elementary treatment of the subject, as here outlined, has its advantages.

however, that while the expression rot $\mathbf{s}$ gives the correct tangential component for the rotation of the element, its normal component is twice that of the actual rotation. The rotation about the normal, however, has no effect upon the unit normal to the element.

The expression for $\mathbf{n}'$ may be put in a different form, which is sometimes more convenient. For, in the above transformation, we may write

$$\frac{1}{H}(\mathbf{r}_1 \times \mathbf{s}_2 - \mathbf{r}_2 \times \mathbf{s}_1)$$
$$= \mathbf{n}\left(\frac{1}{E}\mathbf{r}_1 \cdot \mathbf{s}_1 + \frac{1}{G}\mathbf{r}_2 \cdot \mathbf{s}_2\right) - \left[\frac{1}{E}(\mathbf{n} \cdot \mathbf{s}_1)\,\mathbf{r}_1 + \frac{1}{G}(\mathbf{n} \cdot \mathbf{s}_2)\,\mathbf{r}_2\right]$$
$$= \mathbf{n}\,\mathrm{div}\,\mathbf{s} - (\nabla\mathbf{s}) \cdot \mathbf{n},$$

and thus finally $$\mathbf{n}' = \mathbf{n} - (\nabla\mathbf{s}) \cdot \mathbf{n} \qquad (6),$$
where $\nabla\mathbf{s}$ is, of course, a dyadic.

105. Curvatures of the deformed surface. Let us now find the second order magnitudes L', M', N' for the surface S', and thence deduce its first curvature J' and its second curvature K'. Take the form (6) for the unit normal $\mathbf{n}'$, denoting $\nabla\mathbf{s} \cdot \mathbf{n}$ by $\mathbf{e}$. Then, since for any surface $L = -\mathbf{r}_1 \cdot \mathbf{n}_1$, we have

$$L' = -(\mathbf{r}_1 + \mathbf{s}_1) \cdot (\mathbf{n}_1 - \mathbf{e}_1) = L - \mathbf{s}_1 \cdot \mathbf{n}_1 + \mathbf{r}_1 \cdot \mathbf{e}_1.$$

Now

$$\mathbf{r}_1 \cdot \mathbf{e}_1 = \mathbf{r}_1 \cdot \frac{\partial}{\partial u}(\nabla\mathbf{s}) \cdot \mathbf{n} + \mathbf{r}_1 \cdot (\nabla\mathbf{s}) \cdot \mathbf{n}_1 = \mathbf{r}_1 \cdot \frac{\partial}{\partial u}(\nabla\mathbf{s}) \cdot \mathbf{n} + \mathbf{s}_1 \cdot \mathbf{n}_1,$$

the parametric curves being orthogonal on S. Hence

$$L' = L + \mathbf{r}_1 \cdot \frac{\partial}{\partial u}(\nabla\mathbf{s}) \cdot \mathbf{n},$$

and similarly $$N' = N + \mathbf{r}_2 \cdot \frac{\partial}{\partial v}(\nabla\mathbf{s}) \cdot \mathbf{n}.$$

For M on any surface we have the alternative expressions $-\mathbf{r}_1 \cdot \mathbf{n}_2$ and $-\mathbf{r}_2 \cdot \mathbf{n}_1$. Taking the former we have on S',

$$M' = -(\mathbf{r}_1 + \mathbf{s}_1) \cdot (\mathbf{n}_2 - \mathbf{e}_2) = M - \mathbf{s}_1 \cdot \mathbf{n}_2 + \mathbf{r}_1 \cdot \mathbf{e}_2.$$

Now

$$\mathbf{r}_1 \cdot \mathbf{e}_2 = \mathbf{r}_1 \cdot \frac{\partial}{\partial v}(\nabla\mathbf{s}) \cdot \mathbf{n} + \mathbf{r}_1 \cdot (\nabla\mathbf{s}) \cdot \mathbf{n}_2 = \mathbf{r}_1 \cdot \frac{\partial}{\partial v}(\nabla\mathbf{s}) \cdot \mathbf{n} + \mathbf{s}_1 \cdot \mathbf{n}_2,$$

and therefore $$M' = M + \mathbf{r}_1 \cdot \frac{\partial}{\partial v}(\nabla\mathbf{s}) \cdot \mathbf{n}.$$

Similarly the other form gives

$$M' = M + \mathbf{r}_2 \cdot \frac{\partial}{\partial u} (\nabla \mathbf{s}) \cdot \mathbf{n}.$$

Having found the fundamental magnitudes for the deformed surface, we may calculate its first and second curvatures. Since the parametric curves are orthogonal on S, we have for the *first curvature* of S'

$$\begin{aligned} J' &= \frac{E'N' + G'L' - 2F'M'}{H'^2} \\ &= \frac{(1-2\theta)}{H^2} \left[(E + 2\mathbf{r}_1 \cdot \mathbf{s}_1) \left(N + \mathbf{r}_2 \cdot \frac{\partial}{\partial v} \nabla \mathbf{s} \cdot \mathbf{n} \right) \right. \\ &\quad \left. + (G + 2\mathbf{r}_2 \cdot \mathbf{s}_2) \left(L + \mathbf{r}_1 \cdot \frac{\partial}{\partial u} \nabla \mathbf{s} \cdot \mathbf{n} \right) - 2 (\mathbf{r}_1 \cdot \mathbf{s}_2 + \mathbf{r}_2 \cdot \mathbf{s}_1) M \right]. \end{aligned}$$

Then, as we may neglect small quantities of the second order, this becomes

$$J' = (1 - 2\theta) J + \mathbf{n} \cdot \nabla^2 \mathbf{s} + 2K \nabla^* \cdot \mathbf{s} \quad \text{............(7).}$$

An alternative expression may be deduced by means of the identity (Art. 19)

$$K\nabla^* \cdot \mathbf{s} = J\nabla \cdot \mathbf{s} - \overline{\nabla} \cdot \mathbf{s} = J\theta - \overline{\nabla} \cdot \mathbf{s},$$

in virtue of which (7) is equivalent to

$$J' = J + \mathbf{n} \cdot \nabla^2 \mathbf{s} - 2\overline{\nabla} \cdot \mathbf{s} \quad \text{..................(8).}$$

Similarly the *second curvature* of the deformed surface is given by

$$\begin{aligned} K' &= \frac{L'N' - M'^2}{H'^2} \\ &= \frac{1-2\theta}{H^2} \left[\left(L + \mathbf{r}_1 \cdot \frac{\partial}{\partial u} \nabla \mathbf{s} \cdot \mathbf{n} \right) \left(N + \mathbf{r}_2 \cdot \frac{\partial}{\partial v} \nabla \mathbf{s} \cdot \mathbf{n} \right) \right. \\ &\quad \left. - \left(M + \mathbf{r}_1 \cdot \frac{\partial}{\partial v} \nabla \mathbf{s} \cdot \mathbf{n} \right) \left(M + \mathbf{r}_2 \cdot \frac{\partial}{\partial u} \nabla \mathbf{s} \cdot \mathbf{n} \right) \right] \\ &= (1 - 2\theta) K + K (\nabla^* \cdot \nabla \mathbf{s}) \cdot \mathbf{n} \quad \text{.........................(9),} \end{aligned}$$

on neglecting small quantities of the second order. In virt·· the above identity this may be expressed in the alterna··

$$K' = (1 - 2\theta) K + J\mathbf{n} \cdot \nabla^2 \mathbf{s} - \mathbf{n} \cdot (\overline{\nabla} \cdot \nabla \mathbf{s})$$

106. Modulus of extension. Change

Let $d\mathbf{r}$ and $d\mathbf{r}'$ be corresponding vector e'

lengths ds and ds' respectively. Then, in virtue of (1),

$$d\mathbf{r}' = d\mathbf{r} + d\mathbf{s},$$

and therefore
$$(ds')^2 = (d\mathbf{r}')^2 = (d\mathbf{r})^2 + 2d\mathbf{r}\cdot d\mathbf{s}$$
$$= ds^2\left(1 + 2\frac{d\mathbf{r}}{ds}\cdot\frac{d\mathbf{s}}{ds}\right).$$

Consequently, on taking the square root of both members,

$$ds' = ds\left(1 + \frac{d\mathbf{r}}{ds}\cdot\frac{d\mathbf{s}}{ds}\right) \quad\text{.................(11).}$$

Now $d\mathbf{r}/ds$ is the unit vector $\mathbf{t}$ in the direction of the element $d\mathbf{r}$, and $d\mathbf{s}/ds$ is the derivative of $\mathbf{s}$ in that direction, also expressible as $\mathbf{t}\cdot\nabla\mathbf{s}$. Hence the increase of length per unit length of the element $d\mathbf{r}$ is

$$\frac{ds' - ds}{ds} = \frac{d\mathbf{r}}{ds}\cdot\frac{d\mathbf{s}}{ds} = \mathbf{t}\cdot\nabla\mathbf{s}\cdot\mathbf{t}.$$

This may be called the *modulus of extension* for the direction of $\mathbf{t}$. It will be denoted by e. Thus

$$e = \mathbf{t}\cdot\nabla\mathbf{s}\cdot\mathbf{t} \quad\text{.......................(12).}$$

This modulus is clearly connected with the extension ratio of Art. 95 by the relation

$$\epsilon = 1 + e \quad\text{..........................(13).}$$

And if we introduce the conic

$$\mathbf{R}\cdot\nabla\mathbf{s}\cdot\mathbf{R} = 1 \quad\text{.......................(14),}$$

whose centre is the point P, which is origin of the vector $\mathbf{R}$, it follows from (12) that:

The extension modulus in any direction is equal to the inverse square of the radius of the conic (14) *in that direction.*

Consequently the sum of the extension moduli in two perpendicular directions is equal to the scalar of $\nabla\mathbf{s}$, which is div $\mathbf{s}$. The same result follows from (12) and the definition of divergence. ᐟnce:

ᐧ*m of the extension moduli in two perpendicular directions* ᐟ*s invariant, and equal to the modulus of dilation at that*

ᐟf zero extension are those of the asymptotes of these will be at right angles where div $\mathbf{s}$

vanishes. The directions of greatest and least extension are those of the axes of (14). These are the principal directions of strain, that is to say, the principal directions for the above representation of S on S' (Art. 97).

Consider any curve on S, with unit tangent $\mathbf{t}$ and arc-length s. In virtue of (1) the tangent to the corresponding curve on S' is parallel to the vector

$$\frac{d\mathbf{r}}{ds}+\frac{d\mathbf{s}}{ds}=\mathbf{t}+\mathbf{t}\cdot\nabla\mathbf{s}.$$

But the square of this vector is $1+2e$, to the first order, and its magnitude therefore $1+e$. Consequently the unit tangent $\mathbf{t}'$ to the deformed curve on S' is given by

$$\mathbf{t}'=(1-e)\,\mathbf{t}+\mathbf{t}\cdot\nabla\mathbf{s}\qquad\qquad(15).$$

Let $\mathbf{p}$, $\mathbf{q}$ be unit vectors in any two surface directions on S, and ϕ the inclination of $\mathbf{q}$ to $\mathbf{p}$. The corresponding unit vectors on S' are

$$(1-e_1)\,\mathbf{p}+\mathbf{p}\cdot\nabla\mathbf{s},\quad(1-e_2)\,\mathbf{q}+\mathbf{q}\cdot\nabla\mathbf{s},$$

where e_1 and e_2 are the extension moduli for the directions of $\mathbf{p}$ and $\mathbf{q}$ respectively. The inclination ϕ' of the directions on S' is such that

$$\begin{aligned}\cos\phi'&=\{(1-e_1)\,\mathbf{p}+\mathbf{p}\cdot\nabla\mathbf{s}\}\cdot\{(1-e_2)\,\mathbf{q}+\mathbf{q}\cdot\nabla\mathbf{s}\}\\&=(1-e_1-e_2)\cos\phi+\mathbf{p}\cdot(\nabla\mathbf{s}+\mathbf{s}\nabla)\cdot\mathbf{q}.\end{aligned}$$

If in place of $\mathbf{p}$, $\mathbf{q}$ we take orthogonal unit vectors $\mathbf{a}$, $\mathbf{b}$, the first term is zero, and $\cos\phi'$ is the circular measure ϑ of the small decrease of inclination of the corresponding linear elements due to the strain. Thus:

The decrease ϑ of inclination of the linear elements in the directions of the orthogonal unit vectors $\mathbf{a}$, $\mathbf{b}$ *is given by*

$$\vartheta=\mathbf{a}\cdot(\nabla\mathbf{s}+\mathbf{s}\nabla)\cdot\mathbf{b}\qquad\qquad(16).$$

On the understanding that the direction of $\mathbf{b}$ is obtained by a positive rotation of one right angle about the normal from the direction of $\mathbf{a}$, (16) is equivalent to

$$\vartheta=\mathbf{a}\cdot[(\nabla\mathbf{s}+\mathbf{s}\nabla)\times\mathbf{n}]\cdot\mathbf{a}=-\mathbf{b}\cdot[(\nabla\mathbf{s}+\mathbf{s}\nabla)\times\mathbf{n}]\cdot\mathbf{b}.$$

If then we introduce the conic

$$\mathbf{R}\cdot[(\nabla\mathbf{s}+\mathbf{s}\nabla)\times\mathbf{n}]\cdot\mathbf{R}=1\qquad\qquad(17),$$

whose plane is the tangent plane at P, and whose centre is at P, which is origin of the vector $\mathbf{R}$, it follows that:

The decrease in the angle measured in the positive sense from any linear element on S to the perpendicular linear element is equal to the inverse square of the radius of the conic (17) *in the direction of the former.*

Those perpendicular directions on S which correspond to perpendicular directions on S' are the *principal directions* of strain at the point considered. The *principal lines* of strain are the orthogonal system on S whose directions at any point are the principal directions of strain. From the above theorem it follows that the directions of the asymptotes of (17) are the principal directions of strain at P. This conic is therefore a rectangular hyperbola, whose asymptotes coincide with the axes of (14).

107. Rotation of element of surface. We have seen that the vector rotation of the surface element of S, due to the strain, has a component tangential to the surface equal to the tangential component of rot $\mathbf{s}$ (Art. 104). We shall now prove that the normal component of this rotation is only half the normal component of rot $\mathbf{s}$.

Let $\mathbf{a}$, $\mathbf{b}$ be the unit surface vectors in the principal directions on S. Then the unit surface vector $\mathbf{a}'$ on S' corresponding to $\mathbf{a}$ is, by (15),

$$\mathbf{a}' = (1-e)\,\mathbf{a} + \mathbf{a}\cdot\nabla\mathbf{s}.$$

Since the small angular turning of the element of surface about the normal is the resolved part of $\mathbf{a}'-\mathbf{a}$ in the direction of $\mathbf{b}$, it follows that the normal resolute ω of the rotation of the element has the value expressed by

$$\omega = \mathbf{b}\cdot(\mathbf{a}'-\mathbf{a}) = \mathbf{a}\cdot\nabla\mathbf{s}\cdot\mathbf{b}.$$

Now we have seen that, for the principal directions,

$$\mathbf{a}\cdot(\nabla\mathbf{s}+\mathbf{s}\nabla)\cdot\mathbf{b} = 0.$$

Hence the last equation may be written

$$\begin{aligned}\omega &= \tfrac{1}{2}\,\mathbf{a}\cdot(\nabla\mathbf{s}-\mathbf{s}\nabla)\cdot\mathbf{b}\\ &= \tfrac{1}{2}(\mathbf{a}\cdot\nabla\mathbf{s})\cdot(\mathbf{n}\times\mathbf{a}) + \tfrac{1}{2}(\mathbf{b}\cdot\nabla\mathbf{s})\cdot(\mathbf{n}\times\mathbf{b})\\ &= \tfrac{1}{2}\,\mathbf{n}\cdot[\mathbf{a}\times(\mathbf{a}\cdot\nabla\mathbf{s}) + \mathbf{b}\times(\mathbf{b}\cdot\nabla\mathbf{s})],\end{aligned}$$

showing that $$\omega = \tfrac{1}{2}\,\mathbf{n}\cdot\text{rot}\,\mathbf{s}.$$

Combining the tangential and normal components of the rotation we see that

The whole rotation of the element of surface due to the deformation is represented by the vector

$$\text{rot}\,\mathbf{s} - \tfrac{1}{2}(\mathbf{n} \cdot \text{rot}\,\mathbf{s})\,\mathbf{n}.$$

If we consider a closed curve drawn on the surface S, and the surface integral of the normal rotation $\tfrac{1}{2}\mathbf{n} \cdot \text{rot}\,\mathbf{s}$ over the enclosed region, we have in virtue of the Circulation Theorem (Vol. I, Art. 124)

$$\iint \mathbf{n} \cdot \text{rot}\,\mathbf{s}\,dS = \int_{\circ} \mathbf{s} \cdot d\mathbf{r},$$

which may be expressed:

The circulation of the displacement round any closed curve drawn on the surface is twice the surface integral of the normal rotation taken over the enclosed region.

108. Alternative method. Some of the preceding results may be deduced from the theory of representation of surfaces as given in Arts. 95–98. Thus the formula

$$d\mathbf{r}' = d\mathbf{r} + d\mathbf{s} = d\mathbf{r} + d\mathbf{r} \cdot \nabla\mathbf{s}$$

is of the form $\qquad d\mathbf{r}' = \Phi \cdot d\mathbf{r},$

where $\qquad \Phi = \mathrm{I} + \mathbf{s}\nabla \qquad \text{.......................(18)},$

I being the unit dyadic, and $\mathbf{s}\nabla$ the conjugate of $\nabla\mathbf{s}$.

Consequently $\qquad \Phi_c = \mathrm{I} + \nabla\mathbf{s},$

and, neglecting small quantities of the second order,

$$\Phi_c \cdot \Phi = \mathrm{I} + \nabla\mathbf{s} + \mathbf{s}\nabla \qquad \text{..................(19)}.$$

If ϵ is the extension ratio for the direction $\mathbf{t}$,

$$\epsilon^2 = \mathbf{t} \cdot (\mathrm{I} + \nabla\mathbf{s} + \mathbf{s}\nabla) \cdot \mathbf{t} = 1 + 2\mathbf{t} \cdot \nabla\mathbf{s} \cdot \mathbf{t}.$$

But $\qquad \epsilon^2 = (1+e)^2 = 1 + 2e,$

and therefore $\qquad e = \mathbf{t} \cdot \nabla\mathbf{s} \cdot \mathbf{t}$

as given in (12).

Again, the sum of the squares of the extension ratios for two perpendicular directions is, by (11) of Art. 97,

$$\epsilon_1{}^2 + \epsilon_2{}^2 = (\Psi_c \cdot \Psi)_s = (\nabla\mathbf{r} + \nabla\mathbf{s} + \mathbf{s}\nabla)_s,$$

whence $\qquad (1+e_1)^2 + (1+e_2)^2 = 2 + 2\,\text{div}\,\mathbf{s},$

and therefore $$e_1 + e_2 = \operatorname{div} \mathbf{s}$$
as proved in Art. 106. Similarly the square of the ratio μ of corresponding elements of area on S' and S is given by

$$\mu^2 = (\Psi_c \cdot \Psi)_{2s} = 1 + 2 \operatorname{div} \mathbf{s},$$

so that $$\mu = 1 + \operatorname{div} \mathbf{s},$$
showing that the modulus of dilation θ has the value

$$\theta = \operatorname{div} \mathbf{s}$$

as proved in Art. 103.

Further $$\Phi_c^{-1} \cdot \Phi^{-1} = \mathrm{I} - \nabla \mathbf{s} - \mathbf{s} \nabla.$$
Consequently, by Art. 96, points of S lying on the small circle $(d\mathbf{r})^2 = \eta^2$ with centre at P, correspond to points of S' on the small ellipse

$$d\mathbf{r}' \cdot (\mathrm{I} - \nabla \mathbf{s} - \mathbf{s} \nabla) \cdot d\mathbf{r}' = \eta^2 \quad \text{..............(20)},$$

which is the extension ellipse with centre at the point P'. The reciprocal extension ellipse, with centre at P, is similarly

$$d\mathbf{r} \cdot (\mathrm{I} + \nabla \mathbf{s} + \mathbf{s} \nabla) \cdot d\mathbf{r} = \eta^2 \quad \text{..............(21)},$$

and the theorems of Art. 96 are still applicable. The principal directions of strain are those of the axes of this ellipse, which coincide with the directions of the axes of (14).

109. Differential invariants for the surface S'. The operator ∇' for the surface S' may be calculated by means of (19) of Art. 98. For

$$\begin{aligned} \Psi_c \cdot \Psi &= \nabla \mathbf{r} \cdot \Phi_c \cdot \Phi \cdot \nabla \mathbf{r} \\ &= \nabla \mathbf{r} + \nabla \mathbf{r} \cdot (\nabla \mathbf{s} + \mathbf{s} \nabla) \cdot \nabla \mathbf{r}, \end{aligned}$$

and the uniplanar reciprocal of this dyadic is, to the first order,

$$(\Psi_c \cdot \Psi)^{-1} = \nabla \mathbf{r} - \nabla \mathbf{r} \cdot (\nabla \mathbf{s} + \mathbf{s} \nabla) \cdot \nabla \mathbf{r}.$$

Consequently, in virtue of (18),

$$\Phi \cdot (\Psi_c \cdot \Psi)^{-1} = \nabla \mathbf{r} - \nabla \mathbf{r} \cdot (\nabla \mathbf{s} + \mathbf{s} \nabla) \cdot \nabla \mathbf{r} + \mathbf{s} \nabla,$$

and (19) of Art. 98 then gives

$$\nabla' = \nabla - \nabla \mathbf{r} \cdot (\nabla \mathbf{s} + \mathbf{s} \nabla) \cdot \nabla + (\mathbf{s} \nabla) \cdot \nabla \quad \text{.........(22)}.$$

Or, since $\nabla \mathbf{r}$ is a unit dyadic for vectors tangential to the surface, and an annihilator for vectors normal to the surface, this relation is equivalent to

$$\nabla' = \nabla - (\nabla \mathbf{s}) \cdot \nabla + \mathbf{n}\,\mathbf{n} \cdot (\mathbf{s} \nabla) \cdot \nabla \quad \text{............(23)}.$$

By means of this operator we may calculate the gradient of any scalar point-function ϕ on S', and the divergence and rotation of any vector function $\mathbf{u}$. These may be denoted by

$$\text{grad}'\,\phi = \nabla'\phi, \quad \text{div}'\,\mathbf{u} = \nabla'\cdot\mathbf{u}, \quad \text{rot}'\,\mathbf{u} = \nabla'\times\mathbf{u}.$$

For example, the first curvature J' of S' may be found in this way. We have seen that the unit normal $\mathbf{n}'$ to S' is given by

$$\mathbf{n}' = \mathbf{n} - (\nabla\mathbf{s})\cdot\mathbf{n}.$$

The *first curvature* of S' is then obtained from the formula

$$\begin{aligned} J' &= -\nabla'\cdot\mathbf{n}' \\ &= -[\nabla\mathbf{n} - (\nabla\mathbf{s})\cdot\nabla\mathbf{n} + \mathbf{n}\,\mathbf{n}\cdot(\mathbf{s}\nabla)\cdot\nabla\mathbf{n}]_s + \text{div}\,(\nabla\mathbf{s}\cdot\mathbf{n}), \end{aligned}$$

in virtue of (23). The third term contributes nothing to the scalar of the dyadic, since $\mathbf{n}$ is perpendicular to its derivatives. Expanding the last term we may write the formula

$$\begin{aligned} J' &= -\nabla\cdot\mathbf{n} + (\nabla\mathbf{s}\cdot\nabla\mathbf{n})_s + \mathbf{n}\cdot\nabla^2\mathbf{s} + \nabla\mathbf{s} : \nabla\mathbf{n} \\ &= J + \mathbf{n}\cdot\nabla^2\mathbf{s} - 2\overline{\nabla}\cdot\mathbf{s} \quad \text{..........................}(24), \end{aligned}$$

as found in Art. 105 by a different method.

We may also use the above operator ∇' to calculate the geometrical properties of the curves into which a given *family of curves* on S are deformed. For, if $\mathbf{t}$ is the unit tangent to the given family on S, that to the deformed curves is

$$\mathbf{t}' = (1 - e)\,\mathbf{t} + \mathbf{t}\cdot\nabla\mathbf{s}.$$

The divergence of the family after the strain is then

$$\text{div}'\,\mathbf{t}' = \nabla'\cdot\mathbf{t}',$$

and the geodesic curvature γ' is given by

$$\gamma' = \mathbf{n}'\cdot\text{rot}'\,\mathbf{t}'.$$

The torsion of the geodesic tangent to the curve on S' is found from the formula

$$\tau' = \mathbf{t}'\cdot\text{rot}'\,\mathbf{t}',$$

and the normal curvature of the surface S' in the direction of the curve from

$$\kappa_n' = (\mathbf{t}'\times\mathbf{n}')\cdot\text{rot}'\,\mathbf{t}'.$$

110. Small deformation by conical projection. An interesting particular case is that in which the surface S is deformed

into another, S', by conical projection. Let the centre, O, of projection be taken as origin for the position vector $\mathbf{r}$. Then the displacement $\mathbf{s}$ is parallel to $\mathbf{r}$, and we may write

$$\mathbf{s} = \psi \mathbf{r} \qquad \text{.........................(25)},$$

where ψ is small, and is a point-function on S. The modulus of dilation is then given by

$$\theta = \operatorname{div}(\psi \mathbf{r}) = 2\psi + \mathbf{r} \cdot \nabla \psi \qquad \text{..............(26)},$$

and the modulus of extension for the direction $\mathbf{t}$ by

$$\begin{aligned} e &= \mathbf{t} \cdot \nabla (\psi \mathbf{r}) \cdot \mathbf{t} = \mathbf{t} \cdot (\nabla \psi \mathbf{r} + \psi \nabla \mathbf{r}) \cdot \mathbf{t} \\ &= \psi + (\mathbf{r} \cdot \mathbf{t})(\mathbf{t} \cdot \nabla \psi) \qquad \text{..........................(27)}. \end{aligned}$$

The normal rotation of an element of surface has the value

$$\tfrac{1}{2}\mathbf{n} \cdot \operatorname{rot}(\psi \mathbf{r}) = \tfrac{1}{2}\mathbf{n} \cdot (\nabla \psi \times \mathbf{r}),$$

and the unit normal to S' is

$$\begin{aligned} \mathbf{n}' &= \mathbf{n} - \mathbf{n} \times \operatorname{rot}(\psi \mathbf{r}) = \mathbf{n} - (\mathbf{n} \cdot \mathbf{r}) \nabla \psi \\ &= \mathbf{n} - p \nabla \psi \qquad \text{....................................(28)}, \end{aligned}$$

where p is the perpendicular distance from O to the tangent plane at P, in the sense of the vector $\mathbf{n}$.

The operator ∇' for the surface S' may be calculated from (23). Thus since

$$\nabla \mathbf{s} = \nabla \psi \mathbf{r} + \psi \nabla \mathbf{r},$$

it follows that

$$(\nabla \mathbf{s}) \cdot \nabla = \nabla \psi \mathbf{r} \cdot \nabla + \psi \nabla,$$

and $$\mathbf{n} \cdot (\mathbf{s} \nabla) \cdot \nabla = \mathbf{n} \cdot (\mathbf{r} \nabla \psi + \psi \nabla \mathbf{r}) \cdot \nabla = p (\nabla \psi) \cdot \nabla.$$

Substituting these values in (23) we obtain

$$\nabla' = (1 - \psi) \nabla - \nabla \psi \mathbf{r} \cdot \nabla + p\mathbf{n} (\nabla \psi) \cdot \nabla \text{.........(29)}.$$

The *first curvature* of S' may be calculated from this formula, or, directly from (24). Thus

$$\begin{aligned} J' &= - \nabla' \cdot \mathbf{n}' = - \nabla' \cdot (\mathbf{n} - p \nabla \psi) \\ &= -(1 - \psi) \nabla \cdot \mathbf{n} + \mathbf{r} \cdot \nabla \mathbf{n} \cdot \nabla \psi + \nabla \cdot (p \nabla \psi). \end{aligned}$$

Expanding the last term, and remembering that $p = \mathbf{r} \cdot \mathbf{n}$, we may write the formula as

$$\begin{aligned} J' &= (1 - \psi) J - \mathbf{r} \cdot \overline{\overline{\nabla}} \psi + p \nabla^2 \psi + (\nabla \mathbf{r} \cdot \mathbf{n} + \nabla \mathbf{n} \cdot \mathbf{r}) \cdot \nabla \psi \\ &= (1 - \psi) J - 2\mathbf{r} \cdot \overline{\overline{\nabla}} \psi + p \nabla^2 \psi \qquad \text{.......................(30)}. \end{aligned}$$

The *second curvature* of S' is most readily deduced from (9). On substituting the value $\psi\mathbf{r}$ for $\mathbf{s}$ we obtain, after reduction (cf. Ex. 10 below),

$$K' = (1 - 2\psi - 2\mathbf{r}\cdot\nabla\psi + p\nabla^*\cdot\nabla\psi)\,K \quad\text{.........(31).}$$

111. Small inextensional deformation. The kind of deformation usually considered by writers on Differential Geometry is that in which each element of arc on the surface is unaltered in length by the deformation, which may then be described as *inextensional.* Since the squares of the arc-elements on S and S' are

$$E du^2 + 2F du dv + G dv^2, \quad E' du^2 + 2F' du dv + G' dv^2$$

respectively, in order that these may be equal for all values of du/dv we must have

$$E' = E, \quad F' = F, \quad G' = G,$$

which, in virtue of (2), are equivalent to

$$\left.\begin{array}{c}\mathbf{r}_1\cdot\mathbf{s}_1 = 0, \quad \mathbf{r}_2\cdot\mathbf{s}_2 = 0\\ \mathbf{r}_1\cdot\mathbf{s}_2 + \mathbf{r}_2\cdot\mathbf{s}_1 = 0\end{array}\right\} \quad\text{..................(32).}$$

These are necessary and sufficient conditions that the deformation may be inextensional. The modulus of extension vanishes at every point, and for all surface directions; that is to say

$$\mathbf{t}\cdot\nabla\mathbf{s}\cdot\mathbf{t} = 0 \quad\text{.......................(33)}$$

for every surface vector $\mathbf{t}$.

The conditions (32) may be otherwise expressed. In order that the deformation may be inextensional, the above representation must be conformal, with the extension ratio ϵ identically equal to unity; that is to say, the surfaces must be applicable. By Art. 98 the condition for this may be expressed

$$\Psi_c\cdot\Psi = \nabla\mathbf{r},$$

and therefore, by Art. 109,

$$\nabla\mathbf{r}\cdot(\nabla\mathbf{s} + \mathbf{s}\nabla)\cdot\nabla\mathbf{r} = 0 \quad\text{..................(34).}$$

This dyadic equation expresses the necessary and sufficient conditions. If this relation holds it is evident that (33) is satisfied for all surface directions $\mathbf{t}$; and, in virtue of (16), perpendicular linear elements remain perpendicular.

The modulus of dilation, div $\mathbf{s}$, vanishes identically, being equal

to the sum of the extension moduli for two perpendicular directions. Also

$$\nabla^* \cdot \mathbf{s} = \frac{1}{LN - M^2} [\mathbf{r}_1 \cdot (N\mathbf{s}_1 - M\mathbf{s}_2) + \mathbf{r}_2 \cdot (L\mathbf{s}_2 - M\mathbf{s}_1)] \equiv 0,$$

in virtue of (32). Similarly $\overline{\nabla} \cdot \mathbf{s} = 0$. Thus *for a small inextensional deformation each of the quantities* $\nabla \cdot \mathbf{s}$, $\overline{\nabla} \cdot \mathbf{s}$ *and* $\nabla^* \cdot \mathbf{s}$ *vanishes identically.*

The change in the unit normal retains the same form as in the general case; but the expressions for the *curvatures of the deformed surface* are simplified. Thus (7) and (8) become

$$J' = J + \mathbf{n} \cdot \nabla^2 \mathbf{s} \qquad \text{.....................(35).}$$

In virtue of the identity

$$\text{rot rot } \mathbf{s} = \nabla \text{ div } \mathbf{s} - \nabla^2 \mathbf{s},$$

and the fact that div $\mathbf{s}$ is everywhere zero, (35) is equivalent to

$$J' = J - \mathbf{n} \cdot \text{rot rot } \mathbf{s} \qquad \text{..................(36).}$$

The second curvature of S' is equal to K since, by Gauss's theorem, the value of this curvature is determined by the values of E, F, G and their derivatives of the first two orders.

It was shown in Art. 107 that the angular rotation ω of the surface element about the normal is given by

$$\left.\begin{aligned} \omega &= \mathbf{a} \cdot \nabla \mathbf{s} \cdot \mathbf{b} = -\, \mathbf{b} \cdot \nabla \mathbf{s} \cdot \mathbf{a} \\ &= \tfrac{1}{2} \mathbf{n} \cdot \text{rot } \mathbf{s} \end{aligned}\right\} \qquad \text{...............(37).}$$

This is the negative of the function studied by Weingarten, and called by Bianchi the *characteristic function.* It may be shown to satisfy the differential equation

$$J\omega + \text{div } \nabla^* \omega = 0 \qquad \text{.....................(38),}$$

when the deformation is inextensional†.

***112. Small deformation of a family of surfaces.** Lastly let us consider briefly a small deformation of a singly infinite family of surfaces in space. To fix our ideas suppose that the surfaces are determined by the particles of a material medium occupying the space. The particles of the medium are displaced,

† For a further discussion of small inextensional deformations of a surface see Eisenhart, *Differential Geometry*, Chapter XI, or Forsyth, *Lectures on Differential Geometry*, pp. 394–406.

and the small displacement $\mathbf{s}$ is a point-function in space. All those particles which, before the deformation of the medium, lie on a given surface S will after the deformation lie on another surface S'. Thus S is deformed into S', and similarly for each surface of the family. We may express the properties of the deformation in terms of three-parametric differential invariants for the space occupied by the family of surfaces.

The particle originally at the point $P(\mathbf{r})$ is displaced to the point $P'(\mathbf{r}')$, such that

$$\mathbf{r}' = \mathbf{r} + \mathbf{s}.$$

The particle at the point $Q(\mathbf{r} + d\mathbf{r})$ adjacent to P is displaced to $Q'(\mathbf{r}' + d\mathbf{r}')$ adjacent to P', so that

$$d\mathbf{r}' = d\mathbf{r} + d\mathbf{s} = d\mathbf{r} + d\mathbf{r} \cdot \nabla\mathbf{s} = (\mathrm{I} + \mathbf{s}\nabla) \cdot d\mathbf{r}.$$

Then, by the same reasoning as in the case of a single surface, it follows that particles on the small sphere with centre P and radius η are displaced so as to lie on the *strain ellipsoid*

$$d\mathbf{r}' \cdot (\mathrm{I} - \nabla\mathbf{s} - \mathbf{s}\nabla) \cdot d\mathbf{r}' = \eta^2 \quad \text{..............(39)}$$

with centre at P'; and the extension ratio of the medium for any direction at P varies as the radius of this ellipsoid in the corresponding direction at P'. Similarly, particles on the *reciprocal strain ellipsoid*

$$d\mathbf{r} \cdot (\mathrm{I} + \nabla\mathbf{s} + \mathbf{s}\nabla) \cdot d\mathbf{r} = \eta^2 \quad \text{..............(40)}$$

with centre at P are displaced so as to lie on a small sphere of radius η with centre at P'. The *modulus of extension* at P for the direction of the unit vector $\mathbf{t}$, that is to say, the increase in length per unit length of the linear element of the medium in that direction, is given by

$$e = \mathbf{t} \cdot \nabla\mathbf{s} \cdot \mathbf{t} \quad \text{........................(41).}$$

The *unit normal* for the family of surfaces is now a point-function in space. The unit normal $\mathbf{n}$ at P to the original surface S and the corresponding quantity $\mathbf{n}'$ at P' to the deformed surface S' are connected by the formulae (5) and (6). In terms of three-parametric invariants the latter may be expressed

$$\begin{aligned}\mathbf{n}' &= \mathbf{n} - (\nabla\mathbf{s}) \cdot \mathbf{n} + \mathbf{n}\,(\mathbf{n} \cdot \nabla\mathbf{s}) \cdot \mathbf{n} \\ &= (1 + e)\,\mathbf{n} - (\nabla\mathbf{s}) \cdot \mathbf{n} \quad \text{....................(42).}\end{aligned}$$

where e is the modulus of extension for the direction of $\mathbf{n}$. The unit normal $\bar{\mathbf{n}}$ to the deformed surface through P is therefore

$$\begin{aligned}\bar{\mathbf{n}} &= \mathbf{n}' - \mathbf{s}\cdot\nabla\mathbf{n}' \\ &= (1+e)\,\mathbf{n} - (\nabla\mathbf{s})\cdot\mathbf{n} - \mathbf{s}\cdot\nabla\mathbf{n} \quad\text{............(43).}\end{aligned}$$

Alternative expressions for these quantities may be deduced from (5), which is expressible in terms of three-parametric invariants in the form

$$\begin{aligned}\mathbf{n}' &= \mathbf{n} - \mathbf{n}\times[\operatorname{rot}\mathbf{s} - \mathbf{n}\times(\mathbf{n}\cdot\nabla\mathbf{s})] \\ &= \mathbf{n} - \mathbf{n}\times\operatorname{rot}\mathbf{s} + (\mathbf{n}\cdot\nabla\mathbf{s}\cdot\mathbf{n})\,\mathbf{n} - \mathbf{n}\cdot\nabla\mathbf{s} \\ &= (1+e)\,\mathbf{n} - \mathbf{n}\times\operatorname{rot}\mathbf{s} - \mathbf{n}\cdot\nabla\mathbf{s} \quad\text{..................(44),}\end{aligned}$$

and hence the unit normal to the deformed surface at P is

$$\begin{aligned}\bar{\mathbf{n}} &= \mathbf{n}' - \mathbf{s}\cdot\nabla\mathbf{n}' \\ &= (1+e)\,\mathbf{n} - \mathbf{n}\times\operatorname{rot}\mathbf{s} - \mathbf{n}\cdot\nabla\mathbf{s} - \mathbf{s}\cdot\nabla\mathbf{n} \\ &= (1+e)\,\mathbf{n} - \nabla(\mathbf{n}\cdot\mathbf{s}) + \mathbf{s}\times\operatorname{rot}\mathbf{n} \quad\text{...............(45),}\end{aligned}$$

which may easily be identified with (43).

In virtue of (42) the *first curvature* of the deformed surface at P' is given by

$$\begin{aligned}J' &= -\operatorname{div}\mathbf{n}' = -\operatorname{div}[(1+e)\,\mathbf{n} - (\nabla\mathbf{s})\cdot\mathbf{n}] \\ &= (1+e)\,J - \mathbf{n}\cdot\nabla e + \mathbf{n}\cdot\nabla^2\mathbf{s} + \nabla\mathbf{s} : \nabla\mathbf{n} \quad\text{......(46),}\end{aligned}$$

and that of the deformed surface at P is therefore

$$\begin{aligned}\bar{J} &= J' - \mathbf{s}\cdot\nabla J' \\ &= (1+e)\,J - \mathbf{n}\cdot\nabla e + \mathbf{n}\cdot\nabla^2\mathbf{s} + \mathbf{s}\cdot\nabla\operatorname{div}\mathbf{n} + \nabla\mathbf{s} : \nabla\mathbf{n} \\ &\qquad\text{......(47).}\end{aligned}$$

The modulus θ of *dilation of volume* of the medium at P is the increase in volume per unit volume at that point. By the same argument as in the case of a single surface this modulus may be proved equal to the sum of the extension moduli for three perpendicular directions; and this sum is equal to the scalar of $\nabla\mathbf{s}$, which is $\operatorname{div}\mathbf{s}$. Thus

$$\theta = \operatorname{div}\mathbf{s} \quad\text{..........................(48).}$$

Since the quantity $\mathbf{n}\cdot\operatorname{rot}\mathbf{s}$ has the same value whether $\operatorname{rot}\mathbf{s}$ is the three-parametric invariant for space or the two-parametric invariant for the surface S whose normal is $\mathbf{n}$, it follows that the

normal rotation of the element of the surface S due to the deformation is given by

$$\omega = \tfrac{1}{2}\mathbf{n}\cdot \operatorname{rot}\mathbf{s} \qquad\qquad (49).$$

And, since this is true for every surface S through the point P, it follows that the vector rotation $\mathbf{w}$ of the element of the medium at P due to the deformation has the value†

$$\mathbf{w} = \tfrac{1}{2}\operatorname{rot}\mathbf{s} \qquad\qquad (50).$$

EXAMPLES XI

1. If e_1 and e_2 are the extension moduli for the directions of the unit surface vectors $\mathbf{a}$, $\mathbf{b}$, and ϑ is the change of inclination for linear elements in these directions, then the extension modulus for the direction inclined to $\mathbf{a}$ at an angle θ is given by

$$\begin{aligned} e &= (\mathbf{a}\cos\theta + \mathbf{b}\sin\theta)\cdot\nabla\mathbf{s}\cdot(\mathbf{a}\cos\theta+\mathbf{b}\sin\theta) \\ &= e_1\cos^2\theta + e_2\sin^2\theta + \vartheta\sin\theta\cos\theta. \end{aligned}$$

2. Using two different expansion formulae for the gradient of $\mathbf{n}\cdot\mathbf{s}$ on the surface S, deduce the identity

$$\nabla\mathbf{n}\cdot\mathbf{s} + \nabla\mathbf{s}\cdot\mathbf{n} = \mathbf{s}\cdot\nabla\mathbf{n} + \mathbf{n}\times\operatorname{rot}\mathbf{s},$$

and hence $\qquad \nabla\mathbf{s}\cdot\mathbf{n} = \mathbf{n}\times\operatorname{rot}\mathbf{s}.$

This furnishes another proof of the equivalence of the formulae (5) and (6) for $\mathbf{n}'$.

3. Deduce the formulae (22) and (23) for ∇' from the values of E', F', G', H' found in Art. 103.

4. Deduce the formulae (9) and (10) for K' from the relation $K' = (\nabla'\mathbf{n}')_{2s}$.

5. We have seen that, for a small closed curve drawn on a surface S, and enclosing an element of area dS (Vol. I, Art. 122),

$$\operatorname{div}\mathbf{s} = \mathrm{Lt}\frac{\oint \mathbf{s}\cdot\mathbf{m}\,ds}{dS} - J\mathbf{n}\cdot\mathbf{s},$$

$\mathbf{m}$ being the unit surface vector normal to the curve and directed outward from the enclosed region. From this formula show that $\operatorname{div}\mathbf{s}$ represents the surface dilation due to the small deformation of displacement $\mathbf{s}$.

6. Prove that, in the case of a small inextensional deformation, the normal rotation ω of the element of surface satisfies the differential equation

$$J\omega + \operatorname{div}\nabla^*\omega = 0.$$

† For a fuller treatment of this subject see the author's *Advanced Vector Analysis*, Chapter VIII.

7. Show that, if a small deformation is specified by

$$\mathbf{s} = P\mathbf{r}_1 + Q\mathbf{r}_2 + R\mathbf{n},$$

the conditions that it may be inextensional are expressible as

$$\left.\begin{aligned} 2P_1E + PE_1 + QE_2 &= 2RL \\ P_2E + Q_1G &= 2RM \\ 2Q_2G + QG_2 + PG_1 &= 2RN \end{aligned}\right\}.$$

8. If the orthogonal parametric curves remain orthogonal after the deformation, $\mathbf{r}_1 \cdot \mathbf{s}_2 + \mathbf{r}_2 \cdot \mathbf{s}_1 = 0$. Deduce the relation

$$\mathbf{a} \cdot (\nabla\mathbf{s} + \mathbf{s}\nabla) \cdot \mathbf{b} = 0,$$

and hence show that the principal directions of strain are those of the asymptotes of the conic (17), Art. 106.

9. If $\mathbf{a}$, $\mathbf{b}$ are perpendicular unit surface vectors, then $\mathbf{b} \cdot \nabla\mathbf{s} \cdot \mathbf{n}$ and $-\mathbf{a} \cdot \nabla\mathbf{s} \cdot \mathbf{n}$ represent the rotations of the element of surface about axes parallel to $\mathbf{a}$ and $\mathbf{b}$ respectively. Deduce the tangential component of the rotation of the element.

10. Verify formula (31) of Art. 110.

Substitute $\psi\mathbf{r}$ for $\mathbf{s}$ in (9). Then

$$\begin{aligned} (\nabla^* \cdot \nabla\mathbf{s}) \cdot \mathbf{n} &= [\nabla^* \cdot (\nabla\psi\,\mathbf{r} + \psi\nabla\mathbf{r})] \cdot \mathbf{n} \\ &= [(\nabla^* \cdot \nabla\psi)\,\mathbf{r} + \nabla\psi \cdot \nabla^*\mathbf{r} + \nabla^*\psi \cdot \nabla\mathbf{r} + \psi\,(\nabla^* \cdot \nabla\mathbf{r})] \cdot \mathbf{n}. \end{aligned}$$

Since $\mathbf{n}$ is perpendicular to $\mathbf{r}_1$ and $\mathbf{r}_2$, the second and third terms contribute nothing to the sum. Also $\nabla^* \cdot \nabla\mathbf{r} = 2\mathbf{n}$, and $\mathbf{r} \cdot \mathbf{n} = p$, so that

$$(\nabla^* \cdot \nabla\mathbf{s}) \cdot \mathbf{n} = 2\psi + p\nabla^* \cdot \nabla\psi,$$

while

$$\theta = 2\psi + \mathbf{r} \cdot \nabla\psi.$$

Substituting these values in (9) we obtain (31).

11. For the deformation of Art. 112 show that the small change of inclination ϑ of perpendicular linear elements of the medium parallel to the unit vectors $\mathbf{a}$, $\mathbf{b}$ is

$$\vartheta = \mathbf{a} \cdot (\nabla\mathbf{s} + \mathbf{s}\nabla) \cdot \mathbf{b},$$

and deduce the value $\operatorname{div} \mathbf{s}$ for the modulus of cubical dilation from Gauss's Divergence Theorem.

12. If the deformation of Art. 112 is irrotational ($\operatorname{rot} \mathbf{s} = 0$), show that $\mathbf{s}$ is the gradient of some scalar function ϕ, and hence $\theta = \nabla^2\phi$.

CHAPTER XII

FLEXION OF SURFACES. APPLICABILITY

113. Applicability. Quantities unaltered by flexion*. We have already considered some properties of a small deformation of a surface S, in which the length of each linear element on S is equal to that of the corresponding element on S'. Such inextensional deformation may be called *flexion*, and the surface S' is known as a *deform* of S. A great deal of work has been done in the study of flexion of surfaces by Gauss, Bour, Bonnet, Beltrami, Darboux, Weingarten, Bianchi, Eisenhart and many others. We propose to give a brief account of some of the theorems that have already become classical; and, throughout this chapter, the term deformation will be used in the particular sense of inextensional deformation or flexion.

When a one-to-one correspondence exists between the points of two surfaces, S and S', such that the linear element connecting any two adjacent points on S is equal to that connecting the corresponding points on S', the two surfaces are said to be *applicable*, and the representation of either surface on the other is described as *isometric*. Thus isometric representation is the particular case of conformal representation in which the extension ratio is identically equal to unity. All corresponding infinitesimal figures on the two surfaces are congruent. If the parameters u, v be so chosen that corresponding points are specified by the same parameter values, the square of the linear element

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

has the same form for both surfaces. The lines $u = \text{const.}$ on the two surfaces correspond, and also the lines $v = \text{const.}$; and the first order magnitudes E, F, G have the same values at corresponding points. Now, by means of the Gauss characteristic equation, the

* Throughout this chapter the author is largely indebted to Bianchi, *Geometria Differenziale*, Vol. 1, Chapters VII and VIII, and to Eisenhart, *Differential Geometry*, Chapters VIII and IX.

second curvature K is expressible in terms of E, F, G and their derivatives of the first two orders. It follows that

The second curvature has the same value at corresponding points of two applicable surfaces.

Consequently the Gaussian curvature at any point of a surface is unaltered by flexion of the surface. Moreover, the geodesic curvature of a curve $u = \text{const.}$ has the value $G_1/(2G\sqrt{E})$, and is therefore the same at corresponding points of the applicable surfaces. And, since the parametric curves may be chosen at pleasure, we have the theorem:

The geodesic curvatures of corresponding curves on two applicable surfaces have the same value at corresponding points.

Consequently the geodesic curvature of any curve on a surface is unaltered by flexion of that surface. In particular, if the geodesic curvature vanishes at all points of a curve on S, it vanishes at all points of the corresponding curve on S'. Hence

A geodesic on any surface remains a geodesic when the surface is deformed by flexion.

This theorem is also obvious geometrically from the property that a geodesic through two points is the line of shortest distance on the surface between those points. And, since the geodesic curvature of a curve is the arc-rate of deviation of the curve from the geodesic tangent, it follows, as stated in the previous theorem, that this quantity also remains unaltered by flexion of the surface.

It was shown in Art. 98 that, in order that two surfaces may be applicable it must be possible to establish a correspondence between them in which corresponding vector elements are connected by an equation of the form

$$d\mathbf{r}' = \Phi \cdot d\mathbf{r},$$

the dyadic Φ satisfying the identical relation

$$\nabla\mathbf{r} \cdot \Phi_c \cdot \Phi \cdot \nabla\mathbf{r} = \nabla\mathbf{r}.$$

114. The problem of Minding. The problem of finding necessary and sufficient conditions that two given surfaces be applicable was first proposed by Minding, and is sometimes called the *first problem of applicability*. Let the two surfaces S, S' have

variable second curvatures K, K' and linear elements given by

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2,$$
$$ds'^2 = E'du'^2 + 2F'du'dv' + G'dv'^2,$$

u, v being current parameters on S and u', v' on S'. Then the two surfaces will be applicable provided there exist two independent relations

$$\left.\begin{aligned}\phi\,(u, v) &= \phi'\,(u', v')\\ \psi\,(u, v) &= \psi'\,(u', v')\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

establishing a one-to-one correspondence between the points of the two surfaces, and such that by means of them either of the above expressions for ds^2 and ds'^2 may be transformed into the other. Then the linear elements are equivalent. The curves $\phi = a$ on S will correspond to the curves $\phi' = a$ on S', and the curves $\psi = b$ on the former surface to the curves $\psi' = b$ on the latter. It is evident, therefore, that the gradient $\nabla\phi$ on S must be equal in magnitude to the gradient $\nabla'\phi'$ on S'; and similarly $\nabla\psi$ must be equal in magnitude to $\nabla'\psi'$. Further, the inclination of the curves $\phi = a$ and $\psi = b$ on S is equal to that of the corresponding curves on S'; and, since the gradient $\nabla\phi$ is perpendicular to the curve $\phi = a$, it follows that

$$\nabla\phi \cdot \nabla\psi = \nabla'\phi' \cdot \nabla'\psi'.$$

Thus, in order that the equations (1) may establish the applicability of the two surfaces, it is *necessary* that

$$\left.\begin{aligned}(\nabla\phi)^2 = (\nabla'\phi')^2, \quad (\nabla\psi)^2 = (\nabla'\psi')^2\\ \nabla\phi \cdot \nabla\psi = \nabla'\phi' \cdot \nabla'\psi'\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(2).$$

But these conditions are also *sufficient*. For, if ϕ, ψ are taken as parameters on S, and ϕ', ψ' on S', the squares of the linear elements on the two surfaces are given by (Vol. I, Art. 115)

$$ds^2 = \frac{(\nabla\psi)^2\,d\phi^2 - 2\nabla\phi \cdot \nabla\psi\,d\phi\,d\psi + (\nabla\phi)^2\,d\psi^2}{(\nabla\phi)^2\,(\nabla\psi)^2 - (\nabla\phi \cdot \nabla\psi)^2}$$

and
$$ds'^2 = \frac{(\nabla'\psi')^2 d\phi'^2 - 2\nabla'\phi' \cdot \nabla'\psi'\,d\phi'\,d\psi' + (\nabla'\phi')^2 d\psi'^2}{(\nabla'\phi')^2\,(\nabla'\psi')^2 - (\nabla'\phi' \cdot \nabla'\psi')^2}.$$

Consequently, if the relations (2) hold, corresponding linear elements are equal, and the surfaces are applicable.

To complete the solution of the problem it is necessary to deter-

mine two relations of the form (1). Now we have seen that a necessary condition for the applicability of two surfaces is that their second curvatures have the same value at corresponding points. Consequently

$$K(u, v) = K'(u', v') \quad \text{.....................(3)}$$

may be taken as one of the two relations. The first of the conditions (2) then becomes

$$(\nabla K)^2 = (\nabla' K')^2 \quad \text{.......................(4).}$$

If the equations (3) and (4) are distinct and compatible, they may be taken as the pair of equations (1). Then the surfaces will be applicable provided the other two conditions in (2) are satisfied by

$$\phi = K, \quad \phi' = K', \quad \psi = (\nabla K)^2, \quad \psi' = (\nabla' K')^2.$$

If however all the conditions (2) are not satisfied, the surfaces are not applicable. Also if the relations (3) and (4) are inconsistent, the surfaces are not applicable. This happens, for example, when $(\nabla K)^2$ is a function of K, but $(\nabla' K')^2$ is not the same function of K'.

When, however, the equations (3) and (4) are not independent we may take, in place of (4), the relation

$$\nabla^2 K = \nabla'^2 K' \quad \text{.......................(5),}$$

provided (3) and (5) are independent. Then the necessary and sufficient conditions that (3) and (5) may define an isometric correspondence are expressed by (2), in which

$$\phi = K, \quad \phi' = K', \quad \psi = \nabla^2 K, \quad \psi' = \nabla'^2 K'.$$

The case in which neither (4) nor (5) is independent of (3) is one of some importance, meriting further consideration as in the following Art.

115. Exceptional case. The exceptional case just referred to, in which

$$\left.\begin{aligned} (\nabla K)^2 &= f(K), & (\nabla' K')^2 &= f(K') \\ \nabla^2 K &= F(K), & \nabla'^2 K' &= F(K') \end{aligned}\right\} \quad \text{............(6),}$$

is an interesting one. We proceed to prove the theorem:

When the relations (3) *and* (6) *hold for the two surfaces S and S', these are applicable to each other in a single infinity of ways, being applicable to the same surface of revolution.*

From (6) it is evident that the quotient $\nabla^2 K/(\nabla K)^2$ is a function of K, so that the lines $K=$ const. constitute one family of an isometric system on S (Vol. I, Art. 117). Their orthogonal trajectories, $v=$ const., may be found by quadrature. Now, when the parametric curves $u=$ const. and $v=$ const. are orthogonal, $E=1/(\nabla u)^2$. Also, when they are isometric, it follows from Vol. I, Art. 117, that

$$\frac{\partial}{\partial u}\log\frac{G}{E}=\frac{2\nabla^2 u}{(\nabla u)^2}.$$

Consequently, if K, v are taken as current parameters on the surface, we have, in virtue of (6),

$$\frac{\partial}{\partial K}\log\frac{G}{E}=\frac{2F(K)}{f(K)},$$

so that

$$E=\frac{1}{f(K)},\quad G=\frac{e^{g(K)}}{f(K)},$$

where

$$g(K)=2\int\frac{F(K)\,dK}{f(K)}.$$

The linear element of S is therefore given by

$$ds^2=\frac{1}{f(K)}(dK^2+e^{g(K)}dv^2)\quad\ldots\ldots\ldots\ldots\ldots(7),$$

and that of S' by

$$ds'^2=\frac{1}{f(K')}(dK'^2+e^{g(K')}dv'^2)\quad\ldots\ldots\ldots\ldots\ldots(8).$$

Writing

$$u=\int\frac{dK}{\sqrt{f(K)}}$$

we reduce (7) to the form

$$ds^2=du^2+U^2dv^2\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

where U is a function of u only, depending on the form of the functions $f(K)$ and $F(K)$. But (9) is characteristic of the linear element of a surface of revolution, for which the lines $u=$ const. are parallels. Thus the surfaces S and S', whose linear elements are given by (7) and (8), are applicable to the same surface of revolution, the lines $K=$ const. on either surface corresponding to parallels on the surface of revolution. The equations defining the applicability of S and S' are

$$K=K',\quad v=\pm v'+c,$$

where c is an arbitrary constant, showing that the two surfaces are applicable in an infinite number of ways. The lines $K = a$ on the two surfaces must correspond, but a specified point of one of these lines on S may correspond with any point of the corresponding line on S'.

The above argument shows that any surface, on which the equations

$$(\nabla K)^2 = f(K), \quad \nabla^2 K = F(K) \quad \text{...........(10)}$$

hold, is applicable to a surface of revolution, the parallels of the latter corresponding to the lines $K = \text{const.}$ on the former. Conversely, any surface which is applicable to a surface of revolution has a linear element which is expressible in the form (7). For, as we have already seen, the linear element of such a surface is reducible to the form (9); whence it follows that

$$(\nabla u)^2 = 1, \quad K = -\frac{U''}{U}, \quad \nabla^2 \int \frac{du}{U} = 0 \quad \text{.........(11)}.$$

From the second of these it follows that u is a function of K, or

$$u = \phi(K),$$

and therefore $\int U^{-1} du$ is also a function of K, say

$$\int \frac{du}{U} = \psi(K).$$

Consequently, on differentiating, we have

$$\nabla u = \nabla \phi(K) = \phi'(K) \nabla K \quad \text{.................(i)},$$

and

$$0 = \nabla^2 \int \frac{du}{U} = \nabla^2 \psi(K)$$

$$= \psi''(K)(\nabla K)^2 + \psi'(K)\nabla^2 K \quad \text{...........(ii)},$$

the accents denoting differentiations with respect to K. Squaring (i) we see that $(\nabla K)^2$ is a function of K, and therefore, in virtue of (ii), so also is $\nabla^2 K$. Thus the relations (10) are established, and the linear element is reducible to the form (7). Hence the theorem:

The equations (10) *constitute necessary and sufficient conditions that a surface may be applicable to a surface of revolution.*

Since the equations defining the applicability are

$$K = K', \quad v = \pm v' + c,$$

it follows that every surface, which is applicable to a surface of revolution, admits a continuous deformation into itself in such a way that each curve $K = \text{const.}$ slides over itself*.

116. Surfaces of constant Gaussian curvature. The preceding investigation does not apply to surfaces of constant second curvature. In order that two such surfaces may be applicable to each other, it is clearly necessary that this constant curvature should have the same value for both surfaces. We proceed to show that

Two surfaces of the same constant second curvature are applicable to each other in a triple infinity of ways.

On such a surface, S, let the parametric curves be chosen in the following manner. An arbitrary geodesic, C, is taken as the curve $u = 0$, and the geodesics which cut it orthogonally as the curves $v = \text{const.}$ Then the family of parallels orthogonal to these geodesics may be taken as curves $u = \text{const.}$, since C is one of these parallels. The parameter u may be chosen so that it measures actual distance along a geodesic $v = \text{const.}$; and the parameter v so that it measures actual distance along the curve C from a fixed point $(0, 0)$ as origin of coordinates. The linear element of S has then the geodesic form

$$ds^2 = du^2 + D^2 dv^2 \qquad \text{.....................(12)},$$

where D is such that

$$(D)_{u=0} = 1 \qquad \text{......................(13)},$$

and also, since the geodesic curvature of C is zero,

$$\left(\frac{\partial D}{\partial u}\right)_{u=0} = 0 \qquad \text{......................(14)}.$$

The second curvature of S is given by

$$K = -\frac{1}{D}\frac{\partial^2 D}{\partial u^2} \qquad \text{......................(15)}.$$

Suppose first that $K = 0$. Then (15) gives, on integration,

$$\left.\begin{aligned} D &= uf(v) + F(v) \\ \frac{\partial D}{\partial u} &= f(v) \end{aligned}\right\}.$$

* Eisenhart, *loc. cit.*, p. 326.

The conditions (13) and (14) then show that $f(v) = 0$ and $F(v) = 1$. Consequently $D = 1$, and the linear element has the form

$$ds^2 = du^2 + dv^2.$$

But this is also the linear element of a plane in terms of orthogonal Cartesian coordinates, agreeing with the known property that every developable surface is applicable to a plane.

Suppose next that K is positive and equal to k^2. Then the equation (15) becomes

$$\frac{\partial^2 D}{\partial u^2} + k^2 D = 0,$$

which gives, on integration,

$$\left.\begin{aligned} D &= f(v)\cos ku + F(v)\sin ku \\ \frac{\partial D}{\partial u} &= -kf(v)\sin ku + kF(v)\cos ku \end{aligned}\right\}.$$

The conditions (13) and (14) then show that $F(v) = 0$ and $f(v) = 1$. Consequently the linear element has the form

$$ds^2 = du^2 + \cos^2 ku\, dv^2 \quad \text{..................(16).}$$

But this is also the linear element of a sphere of radius $1/k$. Hence:

Every surface of constant second curvature k^2 is applicable to a sphere of radius $1/k$.

Lastly suppose that K is negative and equal to $-k^2$. Such surfaces of constant negative second curvature are called *pseudospherical surfaces*. The equation (15) then becomes

$$\frac{\partial^2 D}{\partial u^2} = k^2 D,$$

so that

$$\left.\begin{aligned} D &= f(v)\cosh ku + F(v)\sinh ku \\ \frac{\partial D}{\partial u} &= kf(v)\sinh ku + kF(v)\cosh ku \end{aligned}\right\} \quad \text{.........(17).}$$

The conditions (13) and (14) then show that $F(v) = 0$ and $f(v) = 1$, so that $D = \cosh ku$, and the linear element has the value

$$ds^2 = du^2 + \cosh^2 ku\, dv^2 \quad \text{..................(18).}$$

Since the linear element of any pseudospherical surface, of second curvature $-k^2$, is reducible to this form, it follows that

All pseudospherical surfaces of the same Gaussian curvature are applicable to one another.

Thus it has been shown in all cases that two surfaces of the same constant Gaussian curvature are applicable to each other. Moreover, they are applicable in a triple infinity of ways. For, in the above argument, the choice of the origin is arbitrary for each surface, and also the direction of the geodesic $C\,(u=0)$. When these have been chosen arbitrarily for each surface, the linear elements have the same form, and the parametric curves on S correspond to those on S'. The origin on either surface may be chosen in a double infinity of ways, and the geodesic C through it in a single infinity; hence the surfaces are applicable in a triply infinite number of ways.

117. Three forms of ds^2 for pseudospherical surfaces. The form (18) for the linear element of a pseudospherical surface is called the *hyperbolic* form. But there are two other forms for ds^2 of equal importance with the above. If geodesic polar coordinates are chosen, with pole at an ordinary point of the surface, we have (Vol. I, Art. 57)

$$ds^2 = du^2 + D^2 dv^2,$$

with the conditions

$$(D)_{u=0} = 0, \quad \left(\frac{\partial D}{\partial u}\right)_{u=0} = 1 \qquad \text{.............. (19).}$$

If these are applied to (17) we find $f(v)=0$ and $kF(v)=1$. Consequently

$$ds^2 = du^2 + \frac{1}{k^2}\sinh^2 ku\, dv^2 \qquad \text{.............. (20).}$$

This is called the *elliptic* form of ds^2.

The third form may be derived as follows. Let parameters and parametric curves be chosen as in Art. 116, with the single exception that a line of constant geodesic curvature k is taken for the curve $C\,(u=0)$. Then the linear element has the geodesic form (12), and we may write the integral of

$$\frac{\partial^2 D}{\partial u^2} = k^2 D$$

as

$$D = f(v)\, e^{ku} + F(v)\, e^{-ku},$$

so that

$$\frac{\partial D}{\partial u} = kf(v)\, e^{ku} - kF(v)\, e^{-ku}.$$

Then, since v measures actual distance along the curve C, we have

$$(D)_{u=0}=1,$$

and, since the geodesic curvature of C has the value k, it follows that

$$\left(\frac{\partial D}{\partial u}\right)_{u=0}=k.$$

In virtue of these we have

$$\left.\begin{array}{l} f(v)+F(v)=1 \\ f(v)-F(v)=1 \end{array}\right\},$$

so that $f(v)=1$, $F(v)=0$ and $D=e^{ku}$. The linear element is then

$$ds^2=du^2+e^{2ku}dv^2 \quad \text{.....................(21)}.$$

This is the *parabolic* form of ds^2. We may notice that the geodesic curvature of any line $u=\text{const.}$ has the value

$$\frac{1}{D}\frac{\partial D}{\partial u}=k,$$

and is therefore the same for all the parallels $u=\text{const.}$ A line of constant geodesic curvature is called an *oricycle.* Hence the theorem:

On a pseudospherical surface all the curves parallel to an oricycle are themselves oricycles.

Ex. *Pseudospherical surfaces of revolution.* To find the surfaces of revolution which are pseudospherical, of second curvature $-k^2$, take the meridians and parallels as parametric curves, v being the longitude, and u the arc-length of a meridian. Let r be the perpendicular distance of a point on the surface from the axis of revolution, which may be taken as axis of z. Then r is a function of u, and the linear element of the surface is

$$ds^2=du^2+r^2dv^2.$$

Integration of the equation $\qquad \dfrac{\partial^2 r}{\partial u^2}=k^2 r$

gives $\qquad r=Ae^{ku}+Be^{-ku}$ ....(i).

If either A or B is zero, we may make the other equal to unity by adding a constant to u, that is to say, by suitably choosing the line $u=0$. Thus, since the direction of measurement along a meridian is at our disposal, we may write $r=e^{ku}$, so that

$$ds^2=du^2+e^{2ku}dv^2.$$

If ϕ is the inclination of the tangent to the meridian to the axis of revolution,

$$\sin\phi=\frac{dr}{du}=ke^{ku}=kr \quad \text{.............................(ii)},$$

and $$\cos\phi\, d\phi = k\, dr = k \sin\phi\, du.$$

Also $$dz^2 = du^2 - dr^2 = \cos^2\phi\, du^2,$$

so that $$z = \int \frac{\cos^2\phi\, d\phi}{k\sin\phi},$$

and therefore $$kz = \cos\phi + \log\tan\tfrac{1}{2}\phi \quad \text{.........................(iii).}$$

From (ii) and (iii) it follows that the meridian curve is a *tractrix*, with the z-axis as asymptote, the intercept of the tangent between the point of contact and the z-axis being constant and equal to $1/k$. The parallels of the surface have constant geodesic curvature k.

If A and B in (i) have the same sign, we may make them equal by adding a constant to u. Thus r may be reduced to the form

$$r = c\cosh ku,$$

and since $$dz^2 = du^2 - dr^2,$$

we have $$z = \int \sqrt{1 - c^2k^2\sinh^2 ku}\, du \quad \text{.......................(iv).}$$

Similarly when A and B have opposite signs we may reduce r to the form

$$r = c\sinh ku,$$

whence $$z = \int \sqrt{1 - c^2k^2\cosh^2 ku}\, du \quad \text{.......................(v).}$$

In the last two cases the evaluation of z in terms of u or r requires the use of elliptic functions*.

Thus with the meridians and parallels as parametric curves, the linear element of a pseudospherical surface of revolution assumes the elliptic, hyperbolic or parabolic type. The surface of revolution of a tractrix about its asymptote is called the pseudospherical surface of revolution of the *parabolic* type. Similarly the pseudospherical surfaces of revolution corresponding to (iv) and (v) are said to be of the *hyperbolic* and *elliptic* types respectively. On the former the geodesic curvature of the parallels is equal to $k\tanh ku$, and on the latter $k\coth ku$. The parallels are therefore oricycles.

118. Surface deformable on itself. We have seen that two surfaces of the same constant second curvature are applicable to each other in a triply infinite number of ways. In particular any surface of constant K is deformable on itself. Let us now enquire if a surface of variable K is deformable on itself. It is clear at the outset that *each surface, which is applicable to a surface of revolution, admits a continuous flexion on itself.* For the surface of revolution may slide over itself by rotation about its axis. That the converse is true will be proved immediately.

* For a fuller treatment of pseudospherical surfaces of revolution see Bianchi, *loc. cit.*, pp. 338–340, or Eisenhart, *loc. cit.*, pp. 272–277.

Since a first condition of applicability of two surfaces is that they must have the same second curvature at corresponding points, it is evident that, if a surface is deformable over itself, each line C which slides over itself is a line $K = \text{const.}$ But, when this motion takes place, each line parallel to C also slides over itself, and must therefore be a line $K = \text{const.}$ Again, since the geodesic curvature of any curve on the surface is unaltered by flexion of the surface, it follows that each of the above lines $K = \text{const.}$ must be a line of constant geodesic curvature. Consequently a surface, which is capable of continuous deformation over itself, must possess a family of parallel curves $K = \text{const.}$, each of which is an oricycle. We proceed to establish the converse theorem:

A surface, which possesses a family of parallel oricycles, is applicable to a surface of revolution, whose parallels correspond to the family of oricycles.

On the given surface S let the family of parallel oricycles be taken as the curves $u = \text{const.}$, and their orthogonal geodesics as $v = \text{const.}$ Then the linear element has the geodesic form

$$ds^2 = du^2 + D^2 dv^2.$$

Since each line $u = \text{const.}$ is one of constant geodesic curvature we have

$$\frac{1}{D}\frac{\partial D}{\partial u} = f(u),$$

whence

$$D = UV,$$

where U is a function of u only, and V a function of v only. Then, taking $\int V dv$ as a new parameter v', we may write

$$ds^2 = du^2 + U^2 dv'^2,$$

which is also the linear element for a surface of revolution, whose parallels are the curves $u = \text{const.}$ Consequently the surface S is applicable to a surface of revolution, and therefore admits a continuous deformation over itself.

An excellent illustration is afforded by helicoidal surfaces. We have already defined a helicoid as the surface generated by a curve, which is rotated about a fixed axis and, at the same time, translated in the direction of the axis with a velocity proportional to the angular velocity of rotation. Each point of the generating

curve describes a circular helix. Now, from the mode of generation of the helicoid, it is evident that the surface admits a continuous deformation by sliding over itself. Hence the theorem due to Bour:

A helicoidal surface is applicable to a surface of revolution, the circular helices on the former corresponding to the parallels on the latter.

A surface of revolution and a cylinder are particular cases of helicoidal surfaces, for which the velocity of translation and the angular velocity of rotation respectively vanish.

Ex. 1. Verify that the right helicoid (Vol. I, Art. 26, Ex. 1) is applicable to the catenoid of revolution (obtained by rotating the common catenary about its directrix).

Ex. 2. The generating curve of a helicoid is a straight line cutting the axis at a constant angle. Show that the surface is applicable to an hyperboloid of revolution of one sheet (Bianchi).

Ex. 3. *Dini's pseudospherical helicoid* is a helicoid whose meridians are lines of curvature. To find its generating curve, write the coordinates of the current point in the form (Vol. I, p. 65)

$$x = u\cos v, \quad y = u\sin v, \quad z = f(u) + cv.$$

Show that the meridian plane is inclined to the normal to the surface at an angle α, given by

$$\cos\alpha = \frac{c}{\sqrt{u^2 + c^2 + u^2 f'^2(u)}}.$$

If α is constant, the meridians are lines of curvature. Hence the function $f(u)$ is determined from the equation

$$u^2[1 + f'^2(u)] = c^2\tan^2\alpha,$$

which shows that the length of the tangent to the meridian intercepted between the point of contact and the axis of revolution is $c\tan\alpha$. Consequently the meridian is a tractrix.

Show that the surface is pseudospherical, with Gaussian curvature

$$K = -\frac{1}{c^2}\cos^2\alpha.$$

119. Second problem of applicability. The determination of all surfaces, which are applicable to a given surface S, is known as the second problem of applicability. A complete solution of this problem has not yet been found. The second order magnitudes L, M, N of a surface with a given linear element

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

satisfy the Gauss characteristic equation and the two Mainardi-Codazzi relations. And, in virtue of Bonnet's theorem (Vol. I, Art. 44), each solution of these equations determines a surface applicable to the given surface. The problem may, however, be reduced to the integration of a single partial differential equation of the second order in the following manner.

If x, y, z are the Cartesian coordinates of a point $\mathbf{r}$ on a given surface, and X, Y, Z the direction cosines of the normal, it follows from Gauss's formulae (Vol. I, Art. 41) that

$$\left.\begin{aligned} z_{11} - lz_1 - \lambda z_2 &= LZ \\ z_{12} - mz_1 - \mu z_2 &= MZ \\ z_{22} - nz_1 - \nu z_2 &= NZ \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(22).$$

Let $\mathbf{k}$ be the unit vector in the direction of the z-axis. Then

$$\nabla z = \nabla (\mathbf{r} \cdot \mathbf{k}) = \nabla \mathbf{r} \cdot \mathbf{k},$$

so that ∇z is the component of $\mathbf{k}$ tangential to the surface. Consequently, if ϕ is the inclination of $\mathbf{n}$ to $\mathbf{k}$,

$$(\nabla z)^2 = \sin^2 \phi = 1 - Z^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(23).$$

From (22) it then follows that

$$\begin{aligned} H^{-2}[(z_{11} - lz_1 - \lambda z_2)(z_{22} - nz_1 - \nu z_2) - (z_{12} - mz_1 - \mu z_2)^2] \\ = H^{-2}(LN - M^2) Z^2 = K[1 - (\nabla z)^2] \quad \ldots(24). \end{aligned}$$

Since the second member of this equation is a differential invariant of z, so also is the first*. The coefficients in the differential equation depend only on E, F, G and their derivatives of the first and second orders; and the equation is satisfied by each of the Cartesian coordinates of the current point on any surface applicable to S. Thus the determination of all the surfaces applicable to S is reduced to the integration of a partial differential equation of the second order of the Monge-Ampère type. The equation (23) imposes a restriction on the functions x, y, z. For it shows that

$$(\nabla z)^2 < 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(25),$$

being equal to unity only at those points of the surface at which the tangent plane is parallel to $\mathbf{k}$. Conversely, it may be shown that†, if z is a solution of the differential equation (24), satisfying

* Some writers denote the first member of (24) by $\Delta_{22} z$.

† Bianchi, Arts. 133, 134, or Eisenhart, Art. 138.

the condition (25), it is one of the rectangular coordinates of the current point on a surface applicable to S. In terms of this solution the second order magnitudes of the surface are given by (22), or

$$L = \frac{z_{11} - lz_1 - \lambda z_2}{\sqrt{1 - (\nabla z)^2}} \quad \text{.....................(26)},$$

and similar formulae for M and N.

120. Assigned deform of a given curve on S. Consider now the deformations (if any) of a given surface S, such that a given curve C traced on it is deformed into a given curve C' in space. Let the parametric curves on S be chosen as an orthogonal system, of which the given curve C is the curve $v=0$; and let the arc-length of C be taken as parameter u, so that $E=1$ along C. The same conditions hold on the deformed surface S'. The geodesic curvature, γ, of C is unaltered by flexion of the surface, and is therefore equal to the resolved part of the vector curvature of C' tangential to S'. Thus, if κ is the circular curvature of C', and ϖ the inclination of the principal normal of C' to the normal to S', we have

$$\gamma = \kappa \sin \varpi \quad \text{......................(27)}.$$

Consequently a necessary condition that the deformation be possible is that the circular curvature of C' be at least as great as the geodesic curvature of C on S.

Now the values of γ and κ are known; so that (27) determines ϖ, and therefore also the unit normal $\mathbf{n}$ to S' along C', since this must lie in the normal plane to C'. Also, along this curve, $\mathbf{r}_1$ is the unit tangent to the curve, and is known. Consequently $\mathbf{n} \times \mathbf{r}_1$, which is the unit vector $\mathbf{r}_2/\sqrt{G}$ tangential to the curve $u=$ const., is known along C'. Thus the values of x_1, y_1, z_1 and x_2, y_2, z_2 are known for $v=0$. Differentiating these with respect to u, we obtain the values of x_{11}, y_{11}, z_{11} and x_{12}, y_{12}, z_{12} for $v=0$; and the first two of the equations (22) then determine L and M along C'. Also, since K is unaltered by flexion of the surface, its value is known; and the relation

$$H^2K = LN - M^2$$

then determines N along C', except in the case when $L=0$. Excluding this exceptional case for the present, we know the value of N along C'; and the three equations corresponding to the

last of (22) then give the values of x_{22}, y_{22}, z_{22} for $v=0$. Thus all the first and second derivatives of x, y, z are known along the curve C'.

By differentiating these values with respect to u we obtain the derivatives of the third order, except x_{222}, y_{222}, z_{222}. These may be found by differentiating with respect to v the three equations of the form (24), using the values of the known third derivatives. Continuing the process we obtain the values of the derivatives of x, y, z of all orders for $v=0$. The position vector of the current point on S' is then given by the expansion

$$\mathbf{r}=\mathbf{r}_0+(\mathbf{r}_1)_0\,u+(\mathbf{r}_2)_0\,v+\tfrac{1}{2}(\mathbf{r}_{11})_0\,u^2+(\mathbf{r}_{12})_0\,uv+\tfrac{1}{2}(\mathbf{r}_{22})_0\,v^2+\ldots,$$

which is in general convergent. This equation defines a surface S' satisfying the required condition, the curve $v=0$ on S being deformed so as to coincide with C'. Moreover the equation (27) shows that there are two supplementary values of ϖ. Hence the theorem:

It is possible (in two different ways) to deform a surface S in such a manner that a given curve C upon it comes into coincidence with a given curve C', provided the circular curvature of C' at each point is greater than the geodesic curvature of C at the corresponding point.

It remains to consider the exceptional case in which $L=0$. The normal curvature of S' in the direction of C' is then zero, and C' is therefore an asymptotic line on the deformed surface. The torsion of C' must satisfy the relation $\tau^2=-K$, and its circular curvature must be equal to the geodesic curvature of C on S. Consequently C' is determined when C is given, and we have the theorem:

A surface may be deformed so that a given curve on it becomes an asymptotic line on the deform.

When C' has circular curvature γ and torsion $\pm\sqrt{-K}$, so that it is an asymptotic line on S', the above argument shows that the value of N along C' is arbitrary. But for any chosen value of N, the higher derivatives of x, y, z along C' are uniquely determined as above. Thus there is a singly infinite family of surfaces satisfying the prescribed conditions. The surfaces of this family touch one another along C', for the osculating plane of C' is the tangent plane to each surface. In particular, if C is an asymptotic line on S, it

may be taken as the curve C'. Then there is a family of applicable surfaces having this curve as asymptotic line, and touching one another along the curve. Consequently

A surface may be continuously deformed while a given asymptotic line remains rigid, and continues to be an asymptotic line on each deform.

The question, whether a surface may be deformed continuously while a curve C upon it, other than an asymptotic line, remains rigid, must be answered in the negative. For, since κ is fixed and γ is unaltered by flexion of the surface, it follows from (27) that $\sin\varpi$ has the same value for all positions of the surface. Thus ϖ has necessarily one of two supplementary values. The preceding argument then shows that all deforms corresponding to the same value of ϖ are determined by the same value of $\mathbf{r}$, and are therefore coincident. Hence there are only two possible positions for the surface, and not an infinity of positions as required for a continuous deformation. Thus

A surface can be continuously deformed, while a given curve upon it remains rigid, only if the curve is an asymptotic line.

Flexion of Ruled Surfaces

121. Theorems of Bonnet and Beltrami. Let us enquire whether a surface is deformable in such a way that a family of asymptotic lines remains asymptotic on each deform. Suppose that two surfaces, S and S', are applicable with a system of asymptotic lines on S corresponding to a system of asymptotic lines on S'. Let the two families of asymptotic lines on S, and the corresponding lines on S', be taken as parametric curves; and let the curves $v = \text{const.}$ be those that are asymptotic on both surfaces. Then the second order magnitudes for S are such that

$$L = 0, \quad N = 0,$$

while for S' we have

$$L' = 0.$$

And, since K is the same for both surfaces, it follows that

$$M' = \pm M \quad \text{.........................(i).}$$

Taking the Mainardi-Codazzi relation (7) of Art. 43, Vol. I, for both surfaces, we have

$$M_1 = (l - \mu) M$$

and $$M_1' = (l - \mu) M' + \lambda N'.$$

Therefore, in virtue of (i),

$$\lambda N' = 0.$$

The hypothesis $N' = 0$ makes

$$L = L' = 0, \quad N = N' = 0$$

and $$M = \pm M',$$

so that the two surfaces are congruent or symmetric. If, however, $\lambda = 0$ the curves $v =$ const. are geodesics on both surfaces [Vol. I, Art. 47 (5)]. Being also asymptotic lines they are straight, and the surfaces are ruled with their generators in correspondence. In this case there is a family of ruled surfaces applicable to S. For, we have only to satisfy the other Mainardi-Codazzi relation for both surfaces, that is

$$M_2' - N_1' = (\nu - m) M' - \mu N'$$

and $$M_2 = (\nu - m) M,$$

and therefore, in virtue of (i),

$$N_1' = \mu N'.$$

The general solution of this equation is of the form

$$N' = \phi(v) \overline{N} \quad \text{.......................(28)},$$

where $\overline{N}$ is a particular solution and $\phi(v)$ an arbitrary function of v. The appearance of this arbitrary function shows that there is a family of ruled surfaces applicable to S, with the generators in correspondence; and therefore S is capable of continuous deformation satisfying the given conditions. The sign of M is determined by the nature of the ruled surface, being positive for a right-handed and negative for a left-handed surface; and this nature remains constant during the continuous deformation. Thus we have Bonnet's theorem:

A necessary and sufficient condition that a surface admit an applicable surface with one system of asymptotic lines on each in correspondence is that the surface be ruled. And a ruled surface

admits a continuous deformation in which the generators remain straight.

If two applicable surfaces, S and S', have both systems of asymptotic lines in correspondence, we may take these as parametric curves on each. Then

$$L = N = 0, \quad L' = N' = 0$$

and
$$M' = \pm M.$$

Consequently,

If two applicable surfaces have both systems of asymptotic lines in correspondence, the surfaces are congruent or symmetric.

Lastly, from the preceding argument we may deduce Beltrami's theorem, which states that

The deformation of a ruled surface is unique when the generators remain straight, and a given curve C on S becomes asymptotic on the deformed surface.

Let the parametric equation of C be $u = \psi(v)$. This will be an asymptotic line on S' provided it satisfies the differential equation of such lines,

$$2M'du + N'dv = 0,$$

that is
$$2M'du + \phi(v)\,\overline{N}\,dv = 0.$$

This will be so if

$$2M'\psi'(v) + \overline{N}\phi(v) = 0,$$

which gives the unique value

$$\phi(v) = -\frac{2M'\psi'(v)}{\overline{N}}$$

for the function $\phi(v)$. Hence the surface S' is unique, and the theorem is proved.

122. Second theorem of Bonnet. The question naturally presents itself whether it is possible for two ruled surfaces to be applicable in such a way that the generators of either correspond to curves on the other. To answer this question let the parametric curves on the two surfaces correspond, so that $v =$ const. are the generators on S and $u =$ const. those on S'. Then, since the curves $v =$ const. are both geodesics and asymptotic lines on S, we have

$$\lambda = 0, \quad L = 0.$$

Similarly, since the curves $u = \text{const.}$ are both geodesics and asymptotic lines on S',

$$n = 0, \quad N' = 0.$$

The relations $\lambda = n = 0$ show that the parametric curves are geodesics on both surfaces. Also, since K is the same for both surfaces,

$$M' = \pm M.$$

If now we take the Mainardi-Codazzi relations (9) and (10) of Art. 44, Vol. I, for the surfaces S' and S respectively, it follows from the above that

$$\left.\begin{aligned} \frac{\partial}{\partial v}\left(\frac{M}{H}\right) &= -2m\frac{M}{H} \\ \text{and} \qquad \frac{\partial}{\partial u}\left(\frac{M}{H}\right) &= -2\mu\frac{M}{H} \end{aligned}\right\} \qquad (29).$$

But these equations show that there is a surface Σ, applicable to S and S', and such that its second order magnitudes L'', M'', N'' are

$$L'' = N'' = 0, \quad M'' = \pm M.$$

For, in virtue of (29), these satisfy both the Mainardi-Codazzi relations and the Gauss characteristic equation. On the surface Σ the geodesics $u = \text{const.}$ and $v = \text{const.}$ are also asymptotic lines, and are therefore straight. The surface is therefore doubly ruled, and is consequently a quadric. Hence we have Bonnet's theorem:

If two skew surfaces, S and S', are applicable to each other, the generators of S correspond to those of S', unless the surfaces are applicable to a ruled quadric, with the generators of S and those of S' corresponding to different systems of generators on the quadric.

EXAMPLES XII

1. If a surface is capable of continuous deformation on itself it is a helicoid, a cylinder, or a surface of rotation.

2. It is possible, in an infinite number of ways, to deform a surface so that a given curve on it becomes a line of curvature on the deform, provided that, in the case of a ruled surface, the given curve is not an orthogonal trajectory of the generators.

3. In the deformation of a ruled surface, if one generator remain straight they will all remain straight.

4. A ruled surface may be deformed so that the generators become the principal normals of one of their orthogonal trajectories.

5. It is possible, in an infinite number of ways, to deform a ruled surface so that an assigned non-geodesic curve on it becomes plane.

6. The tangent surfaces to all curves which have the same intrinsic equation $\rho=f(s)$ are applicable in such a way that points on the curve determined by the same value of s correspond.

7. Show that the helicoidal minimal surfaces are applicable to the catenoid and to the right helicoid.

8. If a conjugate system on a surface S corresponds to a conjugate system on more than one surface applicable to S, it corresponds to a conjugate system on an infinity of surfaces applicable to S.

9. To any given ruled surface there is another ruled surface applicable in such a way that the corresponding generators are parallel and in the same sense.

10. A surface applicable to a surface of revolution, with the lines of curvature of the two surfaces in correspondence, is a surface of revolution.

11. A ruled surface can be deformed into another ruled surface in such a way that a geodesic becomes a straight line.

12. When an hyperboloid of revolution of one sheet is deformed into another ruled surface its principal circular section becomes a Bertrand curve, and the generators are parallel to the corresponding binormals of the conjugate Bertrand curve.

13. A ruled surface formed by the binormals of a curve C of variable torsion can be deformed into a right conoid; or, when the torsion of C is constant, into the right helicoid.

14. The only real ruled surfaces that can be deformed into surfaces of revolution are the hyperboloid of revolution of one sheet and the minimal helicoid.

15. When a ruled surface admits a continuous deformation over itself, the second curvature of the surface is constant along the line of striction, the generators cut this line at a constant angle, and the parameter of distribution is constant.

16. Show that the surface of translation

$$x=a(\cos u+\cos v),\quad y=a(\sin u+\sin v),\quad z=c(u+v),$$

is applicable to a surface of revolution.

17. Each sheet of the centro-surface of a pseudospherical surface is applicable to the catenoid.

CHAPTER XIII

CURVILINEAR CONGRUENCES

123. Introduction. The method of treatment of a congruence of curves explained in Vol. I, Arts. 103–105, in which the curves are defined as the curves of intersection of two families of surfaces,

$$f(x, y, z, u, v) = 0, \quad g(x, y, z, u, v) = 0,$$

is not very effective. We proceed to show how the examination of the properties of a curvilinear congruence may be carried much further by two other methods, which differ mainly in the analysis employed. The first of these is suggested by the usual treatment of a rectilinear congruence, the position vector of a point in the space occupied by the congruence being expressed as a function of three parameters u, v, s, the first two of which determine the curve through the point, and the last the position of the point on the curve. This method will serve as an introduction to the second, which is the more powerful.

FIRST METHOD*

124. Notation. Let a surface S, which cuts all the curves of the congruence, be taken as *director surface* or surface of reference. Then any convenient curvilinear coordinates u, v on this surface may be taken as the parameters determining a curve of the congruence. Let s be the length of the curve measured from S, in a specified sense along the curve, to the point whose position vector is $\mathbf{r}$. Then $\mathbf{r}$ is a function of the three parameters u, v, s. Let suffixes 1, 2, 3 denote partial differentiations with respect to u, v, s respectively. Thus

$$\mathbf{r}_1 = \frac{\partial \mathbf{r}}{\partial u}, \quad \mathbf{r}_2 = \frac{\partial \mathbf{r}}{\partial v}, \quad \mathbf{r}_3 = \frac{\partial \mathbf{r}}{\partial s}.$$

* Given by the author in a paper "On Congruences of Curves," *Tôhoku Mathematical Journal*, Vol. 28 (1927), pp. 114–125, read before the *London Math. Soc.*, November, 1925.

Then $\mathbf{r}_3$ is the unit tangent to the curve at the point $\mathbf{r}$. It will be denoted by $\mathbf{t}$. Similarly we shall write

$$\mathbf{t}_1 = \frac{\partial \mathbf{t}}{\partial u} = \mathbf{r}_{31}, \quad \mathbf{t}_2 = \frac{\partial \mathbf{t}}{\partial v} = \mathbf{r}_{32}, \quad \mathbf{t}_3 = \mathbf{r}_{33}.$$

The curves of the congruence are thus the curves of intersection of the two families of surfaces

$$u = \text{const.}, \quad v = \text{const.}$$

In a *congruence of circles* the curvature κ is a function of u, v only, so that $\kappa_3 = 0$. Then, since the torsion is zero, we have

$$\mathbf{t}_{33} = \frac{\partial}{\partial s}(\kappa \mathbf{n}) = \kappa \frac{\partial \mathbf{n}}{\partial s} = -\kappa^2 \mathbf{t}.$$

Thus a congruence of circles is characterised by the relations

$$\left.\begin{aligned} \kappa_3 &= 0 \\ \mathbf{t}_{33} &= -\kappa^2 \mathbf{t} \end{aligned}\right\}.$$

A *surface of the congruence* is determined by a relation between the parameters u, v, say

$$v = \phi(u).$$

Such a surface contains all the curves of the congruence which intersect the curve $v = \phi(u)$ on the director surface. Its properties will be considered later.

125. Foci. Focal surface. Let us consider if a curve (u, v) of the congruence approaches an adjacent curve $(u + du, v + dv)$ at any point or points, so that the shortest distance between the curves is of the second or higher order. Then, for such approximation, the distance from a point $\mathbf{r}(u, v, s)$ on the former curve to a point $\mathbf{r}(u + du, v + dv, s + ds)$ on the latter is of the second or higher order. Hence, as far as quantities of the first order,

$$\mathbf{r}(u, v, s) = \mathbf{r}(u + du, v + dv, s + ds),$$

so that

$$\mathbf{r}_1 du + \mathbf{r}_2 dv + \mathbf{t}\, ds = 0 \quad\text{.....................(1).}$$

Now this is possible only where the three vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{t}$ are coplanar, that is to say where

$$[\mathbf{r}_1, \mathbf{r}_2, \mathbf{t}] \equiv (\mathbf{r}_1 \times \mathbf{r}_2) \cdot \mathbf{t} = 0 \quad\text{.................(2).}$$

This equation determines the *focal surface* of the congruence; and the points in which it is met by any curve of the congruence are

the *foci* of that curve. In terms of the Cartesian coordinates x, y, z of the point $\mathbf{r}$, the equation of the focal surface may be expressed

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2').$$

The number of foci on any curve is the number of roots of (2) regarded as an equation in s. This number depends on the nature of $\mathbf{r}$ as a function of s. If, for instance, $\mathbf{r}$ is a polynomial in s of degree n, the equation (2) is of degree $3n-1$ in s, and there are $3n-1$ foci (real or imaginary) on each curve. In a rectilinear congruence $n=1$, and there are two foci on each ray.

At a point on the focal surface the tangent plane is parallel to the common plane of the three vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{t}$. For, any small displacement $d\mathbf{r}$ on the surface is expressible as $\mathbf{r}_1 du + \mathbf{r}_2 dv + \mathbf{t}\, ds$, which is clearly parallel to this plane. The tangent plane to the focal surface at any point is the *focal plane* at that point. The normal to the focal surface is parallel to the vectors

$$\mathbf{r}_1 \times \mathbf{r}_2, \quad \mathbf{r}_2 \times \mathbf{t}, \quad \mathbf{t} \times \mathbf{r}_1,$$

which are all perpendicular to the focal plane. And, since the tangent $\mathbf{t}$ to a curve at a focus is parallel to the focal plane, we have the theorem:

Any curve of the congruence is tangent to the focal surface at each focus of the curve.

Again, considering any surface $v = \phi(u)$ of the congruence, we see that any small displacement $d\mathbf{r}$ on the surface is given by

$$d\mathbf{r} = \mathbf{r}_1 du + \mathbf{r}_2 \phi'(u)\, du + \mathbf{t}\, ds.$$

At a point on the focal surface this is parallel to the focal plane; so that the focal plane coincides with the tangent plane to the surface $v = \phi(u)$ at the focus considered. Hence:

Any surface of the congruence touches the focal surface at the foci of all its curves.

The value of du/dv giving the adjacent curve which determines a particular focus of the curve (u, v) may be found from (1) in terms of the derivatives of $\mathbf{r}$. For, since $\mathbf{t}$ is perpendicular to each of its derivatives, we have on forming the scalar product of each

member of (1) with $\mathbf{t}_1$, $\mathbf{t}_2$ and $\mathbf{t}_3$ successively,

$$\frac{du}{dv}=-\frac{\mathbf{r}_2\cdot\mathbf{t}_1}{\mathbf{r}_1\cdot\mathbf{t}_1}=-\frac{\mathbf{r}_2\cdot\mathbf{t}_2}{\mathbf{r}_1\cdot\mathbf{t}_2}=-\frac{\mathbf{r}_2\cdot\mathbf{t}_3}{\mathbf{r}_1\cdot\mathbf{t}_3}\quad\text{............(3)},$$

showing incidentally that, at a point on the focal surface,

$$(\mathbf{r}_1\cdot\mathbf{t}_1)(\mathbf{r}_2\cdot\mathbf{t}_2)=(\mathbf{r}_1\cdot\mathbf{t}_2)(\mathbf{r}_2\cdot\mathbf{t}_1)\quad\text{..............(4)}.$$

In the expressions contained in (3) the value of s is that corresponding to the focus considered.

It will tend to brevity of expression in the following argument if we adopt the notation of Chapter V, for oblique curvilinear coordinates. Thus

$$a=\mathbf{r}_1{}^2,\quad b=\mathbf{r}_2{}^2,\quad c=\mathbf{t}^2=1,$$

and $$f=\mathbf{r}_2\cdot\mathbf{t},\quad g=\mathbf{t}\cdot\mathbf{r}_1,\quad h=\mathbf{r}_1\cdot\mathbf{r}_2.$$

Then $$\mathbf{r}_1\cdot\mathbf{t}_1=\tfrac{1}{2}a_3,\quad \mathbf{r}_2\cdot\mathbf{t}_2=\tfrac{1}{2}b_3\quad\text{................(5)},$$

and also, since $\mathbf{t}$ is perpendicular to its derivatives,

$$\mathbf{r}_1\cdot\mathbf{t}_3=g_3,\quad \mathbf{r}_2\cdot\mathbf{t}_3=f_3\quad\text{................(6)}.$$

Similarly it may be verified that

$$\left.\begin{aligned}\mathbf{r}_1\cdot\mathbf{t}_2&=\tfrac{1}{2}(g_2+h_3-f_1)\\ \mathbf{r}_2\cdot\mathbf{t}_1&=\tfrac{1}{2}(f_1-g_2+h_3)\end{aligned}\right\}\quad\text{.................(7)}.$$

The scalar triple product $[\mathbf{r}_1, \mathbf{r}_2, \mathbf{t}]$ is denoted by p, and its square is equal to the determinant D of Art. 40. Similarly $A, B, \ldots, H$ are the cofactors of $a, b, \ldots, h$ respectively in the determinant D.

126. Limits. Limit surface. The theory of rectilinear congruences suggests the examination of the behaviour of common normals to adjacent curves of the congruence. Consider the possibility of a normal at P to the curve (u, v) being also normal to a nearby curve $(u+\delta u, v+\delta v)$ at the neighbouring point P'. Let $\mathbf{r}$ and $\mathbf{r}+\delta\mathbf{r}$ be the position vectors of P and P', corresponding to the parameter values (u, v, s) and $(u+\delta u, v+\delta v, s+\delta s)$ respectively. Then, if $\mathbf{t}$ is the unit tangent to the curve at P, and $\mathbf{t}+\delta\mathbf{t}$ the unit tangent at P', the vector PP' is perpendicular to both $\mathbf{t}$ and $\mathbf{t}+\delta\mathbf{t}$; that is

$$\mathbf{t}\cdot\delta\mathbf{r}=0\quad\text{......................(i)},$$

and $$(\mathbf{t}+\delta\mathbf{t})\cdot\delta\mathbf{r}=0,$$

so that $$\delta\mathbf{t}\cdot\delta\mathbf{r}=0\quad\text{......................(ii)}.$$

From the first of these we have, to the first order,

$$\mathbf{t}\cdot(\mathbf{r}_1\,\delta u+\mathbf{r}_2\,\delta v+\mathbf{t}\,\delta s)=0,$$

whence
$$\delta s=-(g\,\delta u+f\,\delta v) \quad \text{.....................(8).}$$

Similarly, neglecting small quantities of the second and higher orders, we may write (ii)

$$(\mathbf{t}_1\,\delta u+\mathbf{t}_2\,\delta v+\mathbf{t}_3\,\delta s)\cdot(\mathbf{r}_1\,\delta u+\mathbf{r}_2\,\delta v)=0,$$

since $\mathbf{t}$ is perpendicular to its derivatives. On substitution of the value of δs given by (8), this equation becomes, in virtue of (6) and (7),

$$(\tfrac{1}{2}a_3-gg_3)\,\delta u^2+(h_3-fg_3-gf_3)\,\delta u\,\delta v+(\tfrac{1}{2}b_3-ff_3)\,\delta v^2=0.$$

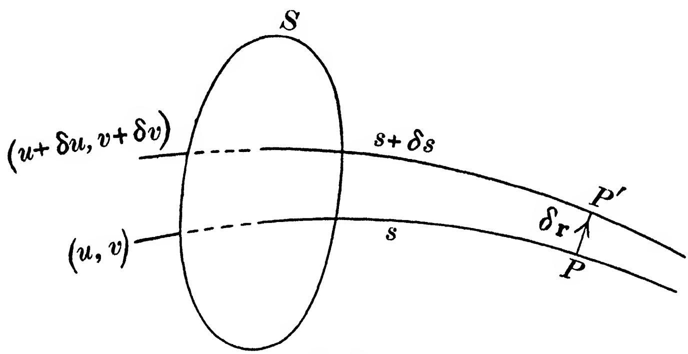

Fig. 10.

Dividing throughout by δv^2, and taking limiting values as δu and δv tend to zero, we have a result which may be expressed

$$B_3\left(\frac{du}{dv}\right)^2-2H_3\,\frac{du}{dv}+A_3=0 \quad \text{..............(9).}$$

From the above argument it follows that the limiting direction of PP' is perpendicular both to $\mathbf{t}$ and to the derivative of $\mathbf{t}$ in this direction. When a direction at a point is perpendicular to the derivative of $\mathbf{t}$ in that direction, we shall describe it as a *direction of zero tendency* for the congruence. Thus the quadratic (9) gives two values of du/dv, which determine the normal directions of zero tendency. Hence:

Of all directions at P normal to the curve through this point, two are also directions of zero tendency for the congruence; and these two directions are determined by the values of du/dv found from (9).

The normal directions determined by (9) vary with the position of P on the curve (u, v). Conversely, for given values of u, v and du/dv, (9) may be regarded as an equation in s determining the foot P of the normal. Thus, on a given curve of the congruence, the point P varies with the value of du/dv. Let us enquire if there are points on the curve at which the foot of the normal is stationary for variation of du/dv. If there is such a point, the coefficients of (9) will be stationary at this point for variation of du/dv. Under these conditions, differentiating (9) with respect to du/dv, we have

$$B_3 \frac{du}{dv} - H_3 = 0 \quad \text{.....................(10).}$$

Substituting this value of du/dv in (9) we find, for such "stationary" points on the curve,

$$A_3 B_3 = H_3^2 \quad \text{.......................(11).}$$

These points correspond to the limit points of a rectilinear congruence; and, for want of a better name, we shall still refer to them as *limit points*, or briefly *limits*, though the term ceases to retain its significance, except in the case of the extreme points. Thus the limits on a given curve (u, v) are determined by the values of s found from (11), for these values of u and v. The equation (11) therefore represents the locus of the limit points, which may be called the *limit surface* of the congruence. Since (11) expresses the condition that (9) should have equal roots in du/dv it follows that, at a limit point on a curve, the two normal directions of zero tendency are coincident. Hence:

On each curve of the congruence there are certain points (called limits) for which the two normal directions of zero tendency are coincident, and the foot of the normal is stationary for variation of these directions.

The number of limits on each curve depends on the nature of $\mathbf{r}$ as a function of s. If $\mathbf{r}$ is a (vector) polynomial of degree n in s, the equation (11) is of degree $4n-2$ in that parameter, and there are $4n-2$ limits on each curve. In the case of a rectilinear congruence $n=1$, and there are two limits on each ray.

Suppose that the value of s for a limit point, as found from (11) in terms of u and v, is substituted in (10). Then for variation of the curve (u, v) this equation may be regarded as the differential

equation of a surface of the congruence which may be called a *principal surface*, as in the case of a rectilinear congruence. Through each curve there pass as many principal surfaces as there are limits on the curve. The tangent plane to a principal surface at the corresponding limit point of a curve C is a *principal plane* for that curve. This principal plane contains the tangent $\mathbf{t}$ to the curve C, and the normal direction of zero tendency.

127. Surface of striction. Divergence of the congruence. In virtue of (8) any small displacement, $\delta\mathbf{r}$, normal to the curve (u, v) at the point $\mathbf{r}$, is expressible in terms of the variations in u and v by

$$\begin{aligned}\delta\mathbf{r} &= \mathbf{r}_1\delta u + \mathbf{r}_2\delta v - (g\,\delta u + f\delta v)\,\mathbf{t}\\ &= (\mathbf{r}_1 - g\mathbf{t})\,\delta u + (\mathbf{r}_2 - f\mathbf{t})\,\delta v.\end{aligned}$$

Consequently, on dividing by δv and taking the limiting value as δv tends to zero, we see that the vector

$$(\mathbf{r}_1 - g\mathbf{t})\frac{du}{dv} + (\mathbf{r}_2 - f\mathbf{t})$$

is normal to the curve at P, for all values of du/dv. Let m, m' represent two different values of du/dv. Then the corresponding normal directions will be at right angles provided

$$(\mathbf{r}_1 - g\mathbf{t})^2\, mm' + (\mathbf{r}_1 - g\mathbf{t})\cdot(\mathbf{r}_2 - f\mathbf{t})\,(m + m') + (\mathbf{r}_2 - f\mathbf{t})^2 = 0,$$

that is $$(a - g^2)\, mm' + (h - fg)\,(m + m') + (b - f^2) = 0,$$

or $$Bmm' - H\,(m + m') + A = 0.$$

If the two normal directions are those of zero tendency, the values of m and m' are the roots of the quadratic (9) in du/dv. Substituting in the above equation the values of the sum and the product of the roots of (9), we see that the two normal directions of zero tendency will be perpendicular to each other provided

$$BA_3 + AB_3 - 2HH_3 = 0 \qquad (12),$$

that is $$\frac{\partial}{\partial s}(AB - H^2) = 0,$$

or $$\frac{\partial D}{\partial s} = 0 \qquad (13).$$

These points of the curve, at which the two normal directions of zero tendency are at right angles, may be called the *points of*

striction of the curve, or its *orthocentres*. The locus of the points of striction of all the curves is the *surface of striction* or *orthocentric surface* of the congruence. Since $D = p^2$, the equation (13) is equivalent to

$$p\frac{\partial p}{\partial s} = 0.$$

It therefore represents the two surfaces

$$p = 0$$

and

$$\frac{\partial p}{\partial s} = 0 \qquad \text{(14)}.$$

The former is the focal surface already considered. The term "surface of striction" will be confined to the locus given by (14). The number of points of striction on any curve is the degree of (14) as an equation in s. If $\mathbf{r}$ is a vector polynomial of degree n in s, this number is $3n - 2$. For a rectilinear congruence $n = 1$, so that there is one orthocentre on each ray. It will be shown later that this is the "middle point" of the ray.

We define the *divergence* of a congruence as the divergence of the unit tangent $\mathbf{t}$. Now it was shown in Art. 43 that the divergence of the vector $X\mathbf{r}_1 + Y\mathbf{r}_2 + Z\mathbf{r}_3$ is equal to

$$\frac{1}{p}\left\{\frac{\partial}{\partial u}(pX) + \frac{\partial}{\partial v}(pY) + \frac{\partial}{\partial s}(pZ)\right\}.$$

Hence the divergence of the congruence has the value

$$\operatorname{div}\mathbf{t} = \frac{1}{p}\frac{\partial p}{\partial s} = \frac{\partial}{\partial s}\log p \qquad \text{(15)}.$$

Since the surface of striction is given by $p_3 = 0$ we have the theorem:

The divergence of the congruence vanishes at all points of the surface of striction.

128. Surfaces of the congruence. Consider the surface of the congruence determined by the relation

$$v = \phi(u) \qquad \text{(16)}.$$

On this surface we may take u, s as parameters since v is now a function of u. The parametric curves $u = \text{const.}$ are the curves of the congruence on the surface; and the curves $s = \text{const.}$ cut the above at a constant arcual distance from the director surface. As

there are now only two independent parameters we must understand "partial derivatives" of $\mathbf{r}$ with respect to u and s as $(\mathbf{r}_1 + \mathbf{r}_2\phi')$ and $\mathbf{r}_3$ respectively. Hence if $\bar{E}$, $\bar{F}$, $\bar{G}$ denote the first order magnitudes for the surface in the Gaussian sense, we have

$$\left.\begin{aligned} \bar{E} &= (\mathbf{r}_1 + \mathbf{r}_2\phi')^2 = a + 2h\phi' + b\phi'^2 \\ \bar{F} &= (\mathbf{r}_1 + \mathbf{r}_2\phi')\cdot\mathbf{r}_3 = g + f\phi' \\ \bar{G} &= \mathbf{r}_3{}^2 = 1 \end{aligned}\right\},$$

and therefore

$$\begin{aligned} V^2 &\equiv \bar{E}\bar{G} - \bar{F}^2 \\ &= (a - g^2) + 2(h - fg)\phi' + (b - f^2)\phi'^2 \\ &= B - 2H\phi' + A\phi'^2. \end{aligned}$$

The unit normal $\mathbf{n}$ to the surface is then

$$\mathbf{n} = \frac{1}{V}(\mathbf{r}_1 + \mathbf{r}_2\phi') \times \mathbf{r}_3,$$

or, inserting the values of the cross products in terms of $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{t}$, as found in Art. 40, we have

$$\mathbf{n} = \frac{1}{pV}[(A\phi' - H)\mathbf{r}_1 + (H\phi' - B)\mathbf{r}_2 + (G\phi' - F)\mathbf{t}] \quad\text{...(17).}$$

In order to find the *first curvature* of the surface, it is easiest to regard the surface as a member of a family of surfaces, say the family defined by

$$\psi(u, v) = \text{const.}$$

Then $$\phi' = \frac{dv}{du} = -\frac{\psi_1}{\psi_2}.$$

The formula (17) for the unit normal then becomes

$$\mathbf{n} = -\frac{(A\psi_1 + H\psi_2)\mathbf{r}_1 + (H\psi_1 + B\psi_2)\mathbf{r}_2 + (G\psi_1 + F\psi_2)\mathbf{t}}{p\sqrt{A\psi_1{}^2 + 2H\psi_1\psi_2 + B\psi_2{}^2}}.$$

Then the first curvature, J, being the negative of the divergence of the unit normal, is given by the formula

$$J = \frac{1}{p}\left[\frac{\partial}{\partial u}\left(\frac{A\psi_1 + H\psi_2}{\kappa}\right) + \frac{\partial}{\partial v}\left(\frac{H\psi_1 + B\psi_2}{\kappa}\right) + \frac{\partial}{\partial s}\left(\frac{G\psi_1 + F\psi_2}{\kappa}\right)\right] \quad\text{......(18),}$$

in which we have written for brevity

$$\kappa = \sqrt{A\psi_1{}^2 + 2H\psi_1\psi_2 + B\psi_2{}^2}.$$

129. Normal congruences. A congruence of curves is said to be normal when it cuts orthogonally a singly infinite family of surfaces. Thus the tangent $\mathbf{t}$ to a curve of the congruence is normal to that surface of the family which passes through the point considered. Let the family of surfaces be given by $\phi = \text{const.}$, where ϕ is some point-function. Then $\mathbf{t}$ is parallel to grad ϕ; or

$$\mu \mathbf{t} = \operatorname{grad} \phi,$$

where μ is also a point-function. Therefore, since the rotation of the gradient vanishes identically (Art. 45),

$$\operatorname{rot} \mu \mathbf{t} = 0,$$

whence
$$\nabla \mu \times \mathbf{t} + \mu \operatorname{rot} \mathbf{t} = 0.$$

Consequently, forming scalar products with $\mathbf{t}$, we have

$$\mathbf{t} \cdot \operatorname{rot} \mathbf{t} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(19).$$

This relation is thus a *necessary* condition that the congruence be normal; and it is shown in works on Differential Equations to be *sufficient* for the existence of a family of surfaces cut orthogonally by the curves of the congruence.

The condition may be neatly expressed in terms of the fundamental magnitudes. For since

$$p \operatorname{rot} \mathbf{t} = \left[\frac{\partial}{\partial v}(\mathbf{t} \cdot \mathbf{r}_3) - \frac{\partial}{\partial s}(\mathbf{t} \cdot \mathbf{r}_2)\right] \mathbf{r}_1 + \left[\frac{\partial}{\partial s}(\mathbf{t} \cdot \mathbf{r}_1) - \frac{\partial}{\partial u}(\mathbf{t} \cdot \mathbf{r}_3)\right] \mathbf{r}_2 + \left[\frac{\partial}{\partial u}(\mathbf{t} \cdot \mathbf{r}_2) - \frac{\partial}{\partial v}(\mathbf{t} \cdot \mathbf{r}_1)\right] \mathbf{r}_3$$

it follows that

$$p\mathbf{t} \cdot \operatorname{rot} \mathbf{t} = -g f_3 + f g_3 + f_1 - g_2.$$

Hence:

A necessary and sufficient condition that the congruence be normal is

$$f g_3 - g f_3 + f_1 - g_2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(20).$$

When the congruence is normal, $\mathbf{t}$ is the unit normal to the surfaces orthogonal to the curves congruence; and the first curvature of a surface of this family is therefore given by

$$J = -\operatorname{div} \mathbf{t} = -\frac{1}{p}\frac{\partial p}{\partial s} \quad \ldots\ldots\ldots\ldots\ldots\ldots(21).$$

The theorem that, in the case of a normal rectilinear congruence, the foci coincide with the limits, may be extended to a normal

congruence of curves. We have seen that, at points on the focal surface (Art. 125),

$$\frac{\mathbf{r}_2 \cdot \mathbf{t}_1}{\mathbf{r}_1 \cdot \mathbf{t}_1} = \frac{\mathbf{r}_2 \cdot \mathbf{t}_2}{\mathbf{r}_1 \cdot \mathbf{t}_2} = \frac{\mathbf{r}_2 \cdot \mathbf{t}_3}{\mathbf{r}_1 \cdot \mathbf{t}_3} \quad \ldots\ldots\ldots\ldots(22).$$

The equality of the first and third of these expressions may be written in terms of the fundamental magnitudes in the form

$$(f_1 - g_2 + h_3)\, g_3 = a_3 f_3.$$

Substituting in the first member of this equation the value of $f_1 - g_2$ given by (20) we have

$$(gf_3 - fg_3 + h_3)\, g_3 = a_3 f_3,$$

that is

$$f_3 B_3 + g_3 H_3 = 0 \quad \ldots\ldots\ldots\ldots(\text{i}).$$

Similarly, from the equality of the second and third of the expressions (22) we find

$$g_3 A_3 + f_3 H_3 = 0 \ldots\ldots\ldots\ldots(\text{ii}).$$

From (i) and (ii) it then follows that, at points on the focal surface,

$$A_3 B_3 = H_3^2.$$

But this is the equation of the limit surface. Hence the theorem*:

For a normal congruence of curves the foci lie on the limit surface.

130. Rectilinear congruences. The theory of rectilinear congruences is a particular case of the above. Along any ray $\mathbf{t}$ is constant, being a function of u and v only. Thus $\mathbf{t}_3 = 0$, and therefore $f_3 = g_3 = 0$. If $\mathbf{a}$ is the position vector of the point in which the ray cuts the director surface, we may write

$$\mathbf{r} = \mathbf{a} + s\mathbf{t} \quad \ldots\ldots\ldots\ldots(23),$$

$\mathbf{a}$ being a function of u and v only. The *focal surface* is given by

$$p = [\mathbf{a}_1 + s\mathbf{t}_1,\ \mathbf{a}_2 + s\mathbf{t}_2,\ \mathbf{t}] = 0,$$

so that, if ρ and ρ' are the distances of the foci from the director surface,

$$\rho + \rho' = -\frac{[\mathbf{a}_1, \mathbf{t}_2, \mathbf{t}] + [\mathbf{t}_1, \mathbf{a}_2, \mathbf{t}]}{[\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}]}.$$

The surface of striction is the surface $p_3 = 0$, that is

$$[\mathbf{t}_1,\ \mathbf{a}_2 + s\mathbf{t}_2,\ \mathbf{t}] + [\mathbf{a}_1 + s\mathbf{t}_1,\ \mathbf{t}_2,\ \mathbf{t}] = 0,$$

showing that there is only one orthocentre on each ray, and that

* Given by the author in the *Trans. Amer. Math. Soc.*, Vol. 31 (1929), p. 128.

the distance of this point from the director surface is equal to $\frac{1}{2}(\rho + \rho')$. Hence:

The orthocentre of a ray coincides with its middle point.

Since $f_3 = g_3 = 0$, the equation (11) of the *limit surface* reduces to

$$a_3 b_3 = h_3^2 \quad \text{.......................(24).}$$

Now $$a = (\mathbf{a}_1 + s\mathbf{t}_1)^2,$$

so that $$a_3 = 2(\mathbf{a}_1 \cdot \mathbf{t}_1 + s\mathbf{t}_1^2).$$

Similarly $$b_3 = 2(\mathbf{a}_2 \cdot \mathbf{t}_2 + s\mathbf{t}_2^2),$$

and $$h_3 = (\mathbf{a}_1 \cdot \mathbf{t}_2 + \mathbf{a}_2 \cdot \mathbf{t}_1) + 2s\,\mathbf{t}_1 \cdot \mathbf{t}_2.$$

Substituting these values in (24) we find, for the equation of the limit surface, a quadratic in s, the sum of whose roots is easily verified to be equal to $\rho_1 + \rho_2$. This is the well-known property that *the point which bisects the distance between the foci also bisects the distance between the limits.*

The condition (20) for a normal congruence becomes, in the case of straight lines,

$$f_1 = g_2.$$

In this case the relations (22), which hold at a focus, give

$$\frac{h_3}{a_3} = \frac{b_3}{h_3},$$

and therefore $$a_3 b_3 = h_3^2.$$

But this is identical with the equation (24) for the limit surface. Hence the well-known theorem that, *in a normal congruence of straight lines, the foci coincide with the limits.*

SECOND METHOD*

131. First quadric. Cone of zero tendency. For a given congruence of curves, the unit vector $\mathbf{t}$, tangent to the curve at any point P, is known as a point-function in space. It has a definite derivative for each direction. In the direction of the unit vector $\mathbf{a}$ its derivative is $\mathbf{a} \cdot \nabla \mathbf{t}$; and the resolved part of this derivative in the direction of $\mathbf{a}$ has the value $\mathbf{a} \cdot \nabla \mathbf{t} \cdot \mathbf{a}$, the operator ∇ being understood to act only on the vector immediately following it. The quantity just defined will be called the *tendency* of the

* This method, as set forth in Arts. 131–139, was given by the author in a paper "On Curvilinear Congruences," *Trans. Amer. Math. Soc.*, Vol. 31 (1929), pp. 117–132.

congruence at P in the direction of $\mathbf{a}$. It plays an important part in the following argument. Denoting it by T we have

$$T = \mathbf{a} \cdot \nabla\mathbf{t} \cdot \mathbf{a} \quad \text{.....................(1).}$$

If then we introduce the quadric surface

$$\mathbf{r} \cdot \nabla\mathbf{t} \cdot \mathbf{r} = 1 \quad \text{.........................(2),}$$

with centre at the point P, which is origin for the vector $\mathbf{r}$, it is clear that *the value of T for any direction at P is equal to the inverse square of the radius of the quadric* (2) *in that direction.* This square may be either positive or negative. Now the sum of the inverse squares of three mutually perpendicular radii of (2) is invariant, being equal to div $\mathbf{t}$, which is the scalar of $\nabla\mathbf{t}$. Hence the sum of the tendencies in three such directions has the value div $\mathbf{t}$. The same result follows from the definition (1), since the sum $\Sigma\mathbf{a} \cdot \nabla\mathbf{t} \cdot \mathbf{a}$ for three mutually perpendicular directions is equal to the scalar of $\nabla\mathbf{t}$ (Art. 68), which is div $\mathbf{t}$. Thus:

The sum of the tendencies of the congruence in three mutually perpendicular directions at a point is invariant, and equal to the divergence of the congruence at that point.

The asymptotic cone of the quadric (2) is given by

$$\mathbf{r} \cdot \nabla\mathbf{t} \cdot \mathbf{r} = 0 \quad \text{.....................(3).}$$

This may be called the *cone of zero tendency* at P; for, in the direction of any of its generators, the tendency of the congruence is zero.

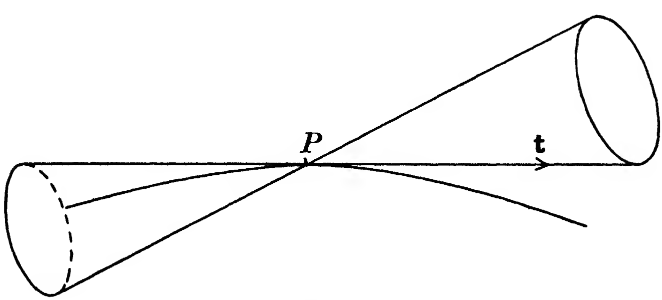

Fig. 11.

The tangent to the curve at P is clearly a generator of this cone; for, since $\mathbf{t}$ is a unit vector, its derivative in the direction of the tangent is perpendicular to $\mathbf{t}$.

We shall be concerned largely with the section of the quadric

(2) by the normal plane of the curve at P, that is to say, by the plane perpendicular to $\mathbf{t}$. This section is the conic

$$\mathbf{r}\cdot\mathbf{t}=0,\quad \mathbf{r}\cdot\nabla\mathbf{t}\cdot\mathbf{r}=1 \quad\text{.................}(4),$$

whose asymptotes are the corresponding section of the cone (3), giving the directions of zero tendency in the normal plane. For a direction inclined at an angle θ in the normal plane to that of a principal axis of the conic (4), the tendency is given by

$$T=T_1\cos^2\theta+T_2\sin^2\theta,$$

where T_1 is the tendency in the direction of the above axis, and T_2 is that for the perpendicular direction. Obviously $T_1+T_2=\operatorname{div}\mathbf{t}$, since the tendency in the direction of $\mathbf{t}$ is zero.

We may observe that a direction of zero tendency in the normal plane corresponds to that of a "common perpendicular" to nearby rays in the case of a rectilinear congruence. This should be borne in mind in order to see the analogy which the following theorems present to those previously known for a congruence of straight lines.

132. Surface of striction. Limit surface. The two directions of zero tendency in the normal plane will be at right angles provided $\operatorname{div}\mathbf{t}$ is zero. This follows from the above theorem on the invariance of the sum of the tendencies in three perpendicular directions. The locus of the points at which this relation holds may be called the *surface of striction* of the congruence, or the *orthocentric surface.* It is analogous to the line of striction of a family of curves on a surface. The points in which it is cut by any curve are the *points of striction* of the curve, or its *orthocentres.* Thus:

The locus of points at which the two directions of zero tendency in the normal plane are at right angles is the surface of striction, which is given by $\operatorname{div}\mathbf{t}=0$.

For a congruence of straight lines the surface of striction is the middle surface, or the locus of points midway between the limits. In the case of a normal congruence of curves the two directions of zero tendency in the normal plane are the asymptotic directions for the surface orthogonal to the curves. These directions are perpendicular when the first curvature of the surface is zero. At such points $\operatorname{div}\mathbf{t}=0$, in agreement with the above.

The *limit surface* of the congruence is the locus of points at which the two directions of zero tendency in the normal plane are coincident. At such points the normal plane touches the cone of zero tendency. Now the normals to the quadric cone $\mathbf{r}\cdot\nabla\mathbf{t}\cdot\mathbf{r}=0$ at its vertex generate another, called the reciprocal cone, whose equation is

$$\mathbf{r}\cdot(\nabla\mathbf{t})^{-1}\cdot\mathbf{r}=0,$$

where $(\nabla\mathbf{t})^{-1}$ is the reciprocal dyadic to $\nabla\mathbf{t}$. But the second of $\nabla\mathbf{t}$ is proportional to the conjugate of $(\nabla\mathbf{t})^{-1}$ (Art. 74). Consequently the equation of the reciprocal cone may also be expressed

$$\mathbf{r}\cdot(\nabla\mathbf{t})_2\cdot\mathbf{r}=0 \quad\text{.......................}(5).$$

Thus, at points on the limit surface, the tangent to the curve must be a generator of the cone (5).

Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be three constant perpendicular unit vectors forming a right-handed system. Let $\mathbf{i}$ have the direction of the tangent $\mathbf{t}$ at the point P. Then $\mathbf{j}$ and $\mathbf{k}$ are parallel to the normal plane of the curve at this point. Let $\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{t}_3$ denote the derivatives of $\mathbf{t}$ in these three directions. Then

$$\nabla\mathbf{t}=\mathbf{i}\,\mathbf{t}_1+\mathbf{j}\,\mathbf{t}_2+\mathbf{k}\,\mathbf{t}_3,$$

and
$$(\nabla\mathbf{t})_2=\mathbf{i}\,(\mathbf{t}_2\times\mathbf{t}_3)+\mathbf{j}\,(\mathbf{t}_3\times\mathbf{t}_1)+\mathbf{k}\,(\mathbf{t}_1\times\mathbf{t}_2).$$

Hence the tangent to the curve will be a generator of the cone (5) provided $\mathbf{t}\cdot(\nabla\mathbf{t})_2\cdot\mathbf{t}=0$, that is

$$\mathbf{t}\cdot(\mathbf{t}_2\times\mathbf{t}_3)=0 \quad\text{.......................}(6).$$

Again, with the same notation, we have

$$\mathbf{t}\operatorname{div}\mathbf{t}-\mathbf{t}\cdot\nabla\mathbf{t}=\mathbf{i}\times(\mathbf{t}\times\mathbf{t}_1)+\mathbf{j}\times(\mathbf{t}\times\mathbf{t}_2)+\mathbf{k}\times(\mathbf{t}\times\mathbf{t}_3),$$

from which it is easily verified that, $\mathbf{t}$ being a unit vector,

$$\begin{aligned}\operatorname{div}(\mathbf{t}\operatorname{div}\mathbf{t}-\mathbf{t}\cdot\nabla\mathbf{t})&=2\mathbf{i}\cdot(\mathbf{t}_2\times\mathbf{t}_3)\\&=2\mathbf{t}\cdot(\mathbf{t}_2\times\mathbf{t}_3).\end{aligned}$$

Thus the condition (6) is equivalent to

$$\operatorname{div}(\mathbf{t}\operatorname{div}\mathbf{t}-\mathbf{t}\cdot\nabla\mathbf{t})=0.$$

Hence the theorem:

The locus of points at which the two directions of zero tendency in the normal plane are coincident is the limit surface, whose equation may be expressed as

$$\operatorname{div}(\mathbf{t}\operatorname{div}\mathbf{t}-\mathbf{t}\cdot\nabla\mathbf{t})=0 \quad\text{.................}(7),$$

or
$$\operatorname{div}(\mathbf{t}\operatorname{div}\mathbf{t}+\mathbf{t}\times\operatorname{rot}\mathbf{t})=0 \quad\text{..............}(7').$$

In the case of a normal congruence of curves, $\mathbf{t}$ is the unit normal to the surface orthogonal to the curves. Then, since the first member of (7) is twice the Gaussian curvature of the surface (Art. 54), it follows that:

The limit surface of a normal congruence is the locus of points at which the second curvature of the orthogonal surfaces is zero.

This agrees with the known property of a surface that the asymptotic directions are coincident where the Gaussian curvature vanishes.

133. Second quadric. Cone of zero moment. Again, let $\mathbf{t}$ be the unit tangent at P, and $\mathbf{t}+\delta\mathbf{t}$ that at an adjacent point Q, such that the vector PQ is $\delta s\,\mathbf{a}$, δs being the length of PQ and $\mathbf{a}$ a unit vector. The mutual moment of the tangents at P and Q, being the resolved part in the direction of $\mathbf{t}$ of the moment of $\mathbf{t}+\delta\mathbf{t}$ about P, has the value

$$\delta s\,\mathbf{a}\times(\mathbf{t}+\delta\mathbf{t})\cdot\mathbf{t}=\delta s\,(\mathbf{a}\times\delta\mathbf{t})\cdot\mathbf{t}.$$

The quotient of this mutual moment by $(\delta s)^2$ has the value $\mathbf{a}\times\dfrac{\delta\mathbf{t}}{\delta s}\cdot\mathbf{t}$; and the limit of this as the point Q tends to coincidence with P, while the direction of $\mathbf{a}$ remains constant, is the function

$$M=\mathbf{a}\times(\mathbf{a}\cdot\nabla\mathbf{t})\cdot\mathbf{t}=\mathbf{a}\cdot(\nabla\mathbf{t}\times\mathbf{t})\cdot\mathbf{a}\;\ldots\ldots\ldots\ldots(8).$$

We shall call this the *moment* of the congruence at P for the direction of $\mathbf{a}$. Let us now introduce the quadric

$$\mathbf{r}\cdot(\nabla\mathbf{t}\times\mathbf{t})\cdot\mathbf{r}=1\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

with centre at P, which is origin for the vector $\mathbf{r}$. Then it is evident from (8) that:

The moment of the congruence for any direction at P is equal to the inverse square of the radius of the quadric (9) *in that direction, having the value zero for directions in the asymptotic cone*

$$\mathbf{r}\cdot(\nabla\mathbf{t}\times\mathbf{t})\cdot\mathbf{r}=0\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10).$$

This cone of zero moment was known to Malus, and has been called by Darboux* the *cone of Malus*. It was found by investigating the directions at P which give nearby points, such that the tangent to the curve at one of these points and the tangent at P have a shortest distance apart of the second or higher order. It is

* *Théorie générale des Surfaces*, T. 2, pp. 262, 280.

the counterpart of the cone (3) of zero tendency. The tangent at P is clearly a generator of the cone of Malus.

Again, it follows from (8) and (9) that the sum of the moments of the congruence in three mutually perpendicular directions at P is invariant. The value of this sum is the scalar of $\nabla\mathbf{t}\times\mathbf{t}$ (Art. 80), which is equal to $\mathbf{t}\cdot\operatorname{rot}\mathbf{t}$. For

$$(\nabla\mathbf{t}\times\mathbf{t})_s=(\mathbf{i}\,\mathbf{t}_1\times\mathbf{t}+\mathbf{j}\,\mathbf{t}_2\times\mathbf{t}+\mathbf{k}\,\mathbf{t}_3\times\mathbf{t})_s$$
$$=(\mathbf{i}\times\mathbf{t}_1\cdot\mathbf{t}+\mathbf{j}\times\mathbf{t}_2\cdot\mathbf{t}+\mathbf{k}\times\mathbf{t}_3\cdot\mathbf{t})=\mathbf{t}\cdot\operatorname{rot}\mathbf{t}.$$

Thus:

The sum of the moments of the congruence for any three mutually perpendicular directions at a point is invariant and equal to $\mathbf{t}\cdot\operatorname{rot}\mathbf{t}$.

This quantity $\mathbf{t}\cdot\operatorname{rot}\mathbf{t}$ may therefore be called the *total moment* of the congruence at P. It vanishes when the congruence is normal, as shown in Art. 129. In this case the cone of Malus has an infinite number of sets of three mutually perpendicular generators.

We shall be concerned largely with the section of the quadric (9) and the cone of Malus by the normal plane at P. The section of the former is the conic

$$\mathbf{r}\cdot\mathbf{t}=0,\quad \mathbf{r}\cdot(\nabla\mathbf{t}\times\mathbf{t})\cdot\mathbf{r}=1 \quad\text{..............(11)},$$

whose asymptotes are the directions of zero moment in the normal plane. For a direction inclined at an angle θ in this plane to a principal axis of the conic (11), the moment is given by

$$M=M_1\cos^2\theta+M_2\sin^2\theta \quad\text{..............(12)},$$

where M_1 is the moment for the direction of the above axis, and M_2 is that for the perpendicular direction. And clearly

$$M_1+M_2=\mathbf{t}\cdot\operatorname{rot}\mathbf{t},$$

since the direction of $\mathbf{t}$ is one of zero moment. It should also be observed that the moment in any direction may be negative. This is the case when the point of the quadric (9) in that direction is imaginary.

We may pause here to draw attention to an interesting relation in connection with rectilinear congruences. It has been shown that, on a ruled surface, the moment M of the family of generators and the second curvature K of the surface at that point are connected by the relation $M=\pm(-K)^{\frac{1}{2}}$ (Art. 28). Consider, then, any

surface of the rectilinear congruence. Let $\mathbf{a}$ be the unit vector which is normal to the ray and tangential to the surface. Then the moment in this direction is $\mathbf{a} \cdot (\nabla \mathbf{t} \times \mathbf{t}) \cdot \mathbf{a}$, and *the second curvature of the ruled surface at P is given by*

$$K = -[\mathbf{a} \cdot (\nabla \mathbf{t} \times \mathbf{t}) \cdot \mathbf{a}]^2.$$

This expression vanishes only when $\mathbf{a}$ is a direction of zero moment; and if this is so for every point P, the surface is a developable surface of the congruence.

134. Surface of normality. Ultimate surface. If the two directions of zero moment in the normal plane are at right angles, the total moment at that point is zero; and conversely. The locus of points at which the congruence possesses this property may be called the *surface of normality*, for at such points the condition is satisfied that the congruence should be normal. It is the surface of zero total moment; and the points in which it is cut by any curve are the points of zero total moment on that curve. Thus:

The locus of points at which the two directions of zero moment in the normal plane are at right angles is the surface of normality, and is given by the equation

$$\mathbf{t} \cdot \operatorname{rot} \mathbf{t} = 0 \qquad \text{(13)}.$$

Let us next consider the locus of points at which the two directions of zero moment in the normal plane are coincident. At such points the normal plane is tangent to the cone of Malus. The condition for the coincidence of the two normal directions of zero moment may be found algebraically with the use of oblique curvilinear coordinates, by the method employed in Art. 126 in the case of the limit surface.

Let P and P' be the points $\mathbf{r}$ and $\mathbf{r} + d\mathbf{r}$, corresponding to the parameter values (u, v, s) and $(u + du, v + dv, s + ds)$ respectively, $\mathbf{t}$ being the unit tangent at P and $\mathbf{t} + d\mathbf{t}$ that at P'. Then, if P' is in the normal plane at P, $d\mathbf{r}$ is perpendicular to $\mathbf{t}$, so that

$$0 = \mathbf{t} \cdot d\mathbf{r} = \mathbf{t} \cdot (\mathbf{r}_1 du + \mathbf{r}_2 dv + \mathbf{r}_3 ds),$$

and therefore $$ds = -(g\,du + f\,dv).$$

Also, if the direction of PP' is one of zero moment,

$$0 = d\mathbf{r} \times (\mathbf{t} + d\mathbf{t}) \cdot \mathbf{t} = (d\mathbf{r} \times d\mathbf{t}) \cdot \mathbf{t}.$$

Expanding the differentials $d\mathbf{r}$, $d\mathbf{t}$ and using the value of ds given above, we may express this condition as

$$0 = \{\mathbf{r}_1 du + \mathbf{r}_2 dv - (g\,du + f\,dv)\,\mathbf{t}\} \cdot \{\mathbf{t}_1 du + \mathbf{t}_2 dv - (g\,du + f\,dv)\,\mathbf{t}_3\} \times \mathbf{r}_3.$$

Then, since the scalar triple products with a repeated factor $\mathbf{t}$ may be disregarded, the two directions du/dv, of zero moment in the normal plane, are given by the quadratic

$$(\mathbf{r}_3 \times \mathbf{r}_1) \cdot (\mathbf{t}_1 - g\mathbf{t}_3) \left(\frac{du}{dv}\right)^2 + \{(\mathbf{r}_3 \times \mathbf{r}_1) \cdot (\mathbf{t}_2 - f\mathbf{t}_3) - (\mathbf{r}_2 \times \mathbf{r}_3) \cdot (\mathbf{t}_1 - g\mathbf{t}_3)\} \frac{du}{dv} - (\mathbf{r}_2 \times \mathbf{r}_3) \cdot (\mathbf{t}_2 - f\mathbf{t}_3) = 0.$$

In terms of the magnitudes A, B, H, etc. of Art. 127, this may be expressed

$$(HB_3 - BH_3 + pBM) \left(\frac{du}{dv}\right)^2 + (BA_3 - AB_3 - 2pHM) \frac{du}{dv} + (AH_3 - HA_3 + pAM) = 0,$$

where M is the total moment of the congruence, given by

$$pM = p\mathbf{t} \cdot \operatorname{rot} \mathbf{t} = f_1 - g_2 + fg_3 - gf_3,$$

as was shown in Art. 129. The condition that the above quadratic in du/dv may have equal roots is reducible to the form

$$\frac{A_3 B_3 - H_3^2}{p^2} - \left(\frac{p_3}{p}\right)^2 + M^2 = 0,$$

or in terms of differential invariants of $\mathbf{t}$,

$$2 \operatorname{div} (\mathbf{t} \operatorname{div} \mathbf{t} - \mathbf{t} \cdot \nabla \mathbf{t}) - (\operatorname{div} \mathbf{t})^2 + (\mathbf{t} \cdot \operatorname{rot} \mathbf{t})^2 = 0 \ldots\ldots(14).$$

Thus we have the theorem:

The locus of points at which the two directions of zero moment in the normal plane are coincident is the surface given by (14).

For convenience we shall refer to this surface as the *ultimate surface*—a name suggested by the term "limit surface" applied in the corresponding case when the two directions of zero tendency are coincident. We may notice what the theorem becomes in the case of a normal congruence. If J and K are the first and second curvatures of the orthogonal surface, the equation (14) expresses that

$$4K - J^2 = 0.$$

That is to say, the ultimate surface of a normal congruence is the locus of the umbilical points of the surfaces orthogonal to the congruence. The interpretation is that at such points the cone of Malus consists of two planes, one of which is perpendicular to $\mathbf{t}$. Any direction in this plane is one of zero moment.

135. Axes of normal sections. We shall now prove the important property that the axes of the conic (4) bisect the angles between those of the conic (11), and *vice versa*. Take rectangular axes of Cartesian coordinates x, y, z so that the first is in the direction of the tangent $\mathbf{t}$ at the point P. Then if $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors in the directions of these axes, $\mathbf{i}$ is parallel to $\mathbf{t}$, while $\mathbf{j}$ and $\mathbf{k}$ are parallel to the normal plane at P. The derivatives of $\mathbf{t}$ in the directions of the coordinate axes may be expressed in the form

$$a\mathbf{j} + a'\mathbf{k}, \quad b\mathbf{j} + b'\mathbf{k}, \quad c\mathbf{j} + c'\mathbf{k}$$

respectively. Then, since $\mathbf{r} = y\mathbf{j} + z\mathbf{k}$, the conic (4) is

$$x = 0, \quad by^2 + (b' + c)\,yz + c'z^2 = 1,$$

and the axes of the conic, being the bisectors of the angles between its asymptotes, are given by

$$x = 0, \quad \frac{y^2 - z^2}{b - c'} = \frac{2yz}{b' + c} \quad \text{.....................(A).}$$

Similarly the conic (11) has for its equations

$$x = 0, \quad b'y^2 + (c' - b)\,yz - cz^2 = 1,$$

and its axes are given by

$$x = 0, \quad \frac{y^2 - z^2}{b' + c} = \frac{2yz}{c' - b} \quad \text{..................(B).}$$

It is clear that the lines (B) are the bisectors of the angles between those given by (A), and *vice versa*. Hence the theorem:

The axes of the section of either of the quadrics (2) *or* (9) *by the normal plane bisect the angles between the axes of the section of the other.*

This includes, as a particular case, the theorem for a rectilinear congruence, that the bisectors of the angles between the focal planes are also the bisectors of the angles between the principal planes. For the cone of Malus is the same at all points of a given ray, consisting of the two focal planes. Thus the planes bisecting

the angles between the focal planes contain the axes of the conic (11) for all points of the ray. Similarly it follows from Hamilton's formula that the principal planes bisect the angles between the asymptotes of the conic (4), and therefore contain the axes of that conic for all points of the ray. From the above theorem it therefore follows that the principal planes are inclined at an angle $\pi/4$ to the bisectors of the angles between the focal planes.

136. Rates of rotation. Consider next the arc-rate at which a direction of zero tendency in the normal plane turns about the tangent as the point P moves along the curve. If $\mathbf{a}$ is the unit vector in this direction of zero tendency we have identically

$$\mathbf{a}\cdot\nabla\mathbf{t}\cdot\mathbf{a}=0 \qquad\qquad (15).$$

Let $\frac{d}{ds}$ denote differentiation along the curve. Then, if $\mathbf{b}$ is the unit vector $\mathbf{t}\times\mathbf{a}$, we may write

$$\frac{d\mathbf{a}}{ds}=\omega\mathbf{b}+\psi\mathbf{t},$$

where ω is the arc-rate of turning about the tangent. Hence on differentiating (15) we have, since $\mathbf{t}$ is perpendicular to its derivatives,

$$(\omega\mathbf{b}+\psi\mathbf{t})\cdot\nabla\mathbf{t}\cdot\mathbf{a}+\mathbf{a}\cdot\frac{d}{ds}(\nabla\mathbf{t})\cdot\mathbf{a}+\omega\mathbf{a}\cdot\nabla\mathbf{t}\cdot\mathbf{b}=0,$$

which may be written

$$\omega\,[\mathbf{b}\cdot(\nabla\mathbf{t}\times\mathbf{t})\cdot\mathbf{b}-\mathbf{a}\cdot(\nabla\mathbf{t}\times\mathbf{t})\cdot\mathbf{a}]=\mathbf{a}\cdot\frac{d}{ds}(\nabla\mathbf{t})\cdot\mathbf{a}+\psi\mathbf{t}\cdot\nabla\mathbf{t}\cdot\mathbf{a} \qquad\qquad (16).$$

At the limit surface of the congruence the coefficient of ω is zero. For $\mathbf{a}$ and $\mathbf{b}$ have then the directions of the axes of the conic (4), and are therefore inclined at equal angles $\pi/4$ to each of the axes of the conic (11). It follows from (12) that the moments in these directions are equal, showing that the coefficient of ω vanishes at the limit surface. To interpret this we observe that, just as ω is the arc-rate of rotation of the direction of zero tendency in the normal plane as the point P moves along the curve, so its reciprocal is the rate at which the point P moves along the curve for rotation of the normal direction of zero tendency. Thus, since the coefficient of ω in (16) vanishes at a limit point, while in

general the second member of (16) does not, we have the following theorem:

At the limit points of a curve the feet of the normals giving the directions of zero tendency in the normal plane are stationary for variation of these directions.

This is substantially the theorem of Art. 126, found by a different method.

Similarly we may examine the rate of rotation of a direction of zero moment in the normal plane. If $\mathbf{a}$ is now the unit vector in such a direction we have identically

$$\mathbf{a} \cdot (\nabla\mathbf{t} \times \mathbf{t}) \cdot \mathbf{a} = 0.$$

Hence, with the same notation as before, we have on differentiation

$$(\omega\mathbf{b} + \psi\mathbf{t}) \cdot (\nabla\mathbf{t} \times \mathbf{t}) \cdot \mathbf{a} + \mathbf{a} \cdot \frac{d}{ds}(\nabla\mathbf{t} \times \mathbf{t}) \cdot \mathbf{a} + \omega\mathbf{a} \cdot (\nabla\mathbf{t} \times \mathbf{t}) \cdot \mathbf{b} = 0,$$

which may be expressed as

$$\omega\left[\mathbf{a} \cdot \nabla\mathbf{t} \cdot \mathbf{a} - \mathbf{b} \cdot \nabla\mathbf{t} \cdot \mathbf{b}\right] = \mathbf{a} \cdot \frac{d}{ds}(\nabla\mathbf{t} \times \mathbf{t}) \cdot \mathbf{a} + \psi\mathbf{t} \cdot (\nabla\mathbf{t} \times \mathbf{t}) \cdot \mathbf{a} \qquad \text{......(17).}$$

At the ultimate surface the coefficient of ω is zero. For $\mathbf{a}$ and $\mathbf{b}$ have then the directions of the axes of the conic (11), and are therefore inclined at equal angles $\pi/4$ to those of the conic (4). It follows that the tendencies in these directions are equal, so that the coefficient of ω in (17) vanishes. In general the second member of (17) is not zero, and we have the following theorem:

At the ultimate points of a curve the feet of the normals giving the directions of zero moment in the normal plane are stationary for variation of these directions.

137. Where the cones are pairs of planes. We have seen that the cone of Malus consists of a pair of planes at all points of a rectilinear congruence, and also at points on the ultimate surface of a normal curvilinear congruence. Let us consider at what points either of the two cones thus becomes a pair of planes. Since the tangent to the curve is a generator of each cone, it is clear that when the cone consists of a pair of planes, one of these must pass through the tangent.

First consider the cone of zero tendency, and suppose that this is a pair of planes. Let $\mathbf{a}$ be the unit vector parallel to the inter-

section of the plane through the tangent with the normal plane at P. Then any direction at P in the plane of $\mathbf{t}$ and $\mathbf{a}$ is one of zero tendency; so that, for all values of ϕ,

$$(\mathbf{t}\cos\phi + \mathbf{a}\sin\phi)\cdot\nabla\mathbf{t}\cdot(\mathbf{t}\cos\phi + \mathbf{a}\sin\phi) = 0.$$

Since $\mathbf{t}$ is perpendicular to its derivatives, the first term in the last factor contributes nothing to the value of the product. Also $\mathbf{a}$ is by hypothesis a direction of zero tendency, so that the above condition requires that

$$\mathbf{t}\cdot\nabla\mathbf{t}\cdot\mathbf{a} = 0,$$

which may be expressed as

$$\kappa\mathbf{n}\cdot\mathbf{a} = 0,$$

where κ is the curvature of the curve and $\mathbf{n}$ the unit principal normal. This equation is satisfied if κ is zero, or if $\mathbf{a}$ is the unit binormal to the curve. Hence:

At points where the curvature of a curve is zero, or where the direction of the binormal is one of zero tendency, the cone of zero tendency consists of a pair of planes.

The former condition is satisfied at all points for a rectilinear congruence; and in this case the cone of zero tendency at any point of a ray is a pair of planes equally inclined to each principal plane.

Next consider the cone of Malus. In order that this may consist of a pair of planes we must have, for all values of ϕ,

$$(\mathbf{t}\cos\phi + \mathbf{a}\sin\phi)\cdot(\nabla\mathbf{t}\times\mathbf{t})\cdot(\mathbf{t}\cos\phi + \mathbf{a}\sin\phi) = 0,$$

$\mathbf{a}$ being the unit vector parallel to the intersection of the normal plane with the plane of zero moment through the tangent. This requires that

$$\mathbf{t}\cdot(\nabla\mathbf{t}\times\mathbf{t})\cdot\mathbf{a} = 0,$$

that is to say,

$$\kappa(\mathbf{n}\times\mathbf{t})\cdot\mathbf{a} = 0.$$

Now $\mathbf{t}\times\mathbf{n}$ is the unit binormal to the curve. Hence, in order that the above condition may hold, either κ must vanish, or $\mathbf{a}$ must be parallel to $\mathbf{n}$. Thus:

At points where the curvature of a curve is zero, or where the direction of the principal normal is one of zero moment, the cone of Malus consists of a pair of planes.

The former condition is satisfied at all points for a rectilinear congruence; and in this case the cone of Malus consists of the two focal planes, which are the same for all points of a given ray.

***138. Isotropic congruences.** The conception of an isotropic congruence of straight lines may be generalised so as to apply to curvilinear congruences. An isotropic rectilinear congruence is one whose limit surface coincides with its surface of striction. Now at points on the former surface the normal plane is tangent to the cone of zero tendency; while at points on the latter the sum of the tendencies in two perpendicular directions in the normal plane is zero. In order that both properties may be possessed simultaneously, the cone of zero tendency must consist of two planes, one of which is the normal plane. Conversely, points at which the cone (3) behaves in this manner must lie on both the limit surface and the surface of striction. We shall therefore define an *isotropic curvilinear congruence* as one for which the normal plane is part of the cone of zero tendency at all points of the surface of striction. This surface may then be described as a limit-striction surface. At points on it the equation (7) holds simultaneously with div $\mathbf{t} = 0$.

Let $\mathbf{a}$ and $\mathbf{b}$ be a pair of perpendicular unit vectors in the normal plane. Then, at points where the above property holds, the direction of the unit vector $\mathbf{a}\cos\phi + \mathbf{b}\sin\phi$ must be one of zero tendency for all values of ϕ. Hence the equation

$$(\mathbf{a}\cos\phi + \mathbf{b}\sin\phi)\cdot\nabla\mathbf{t}\cdot(\mathbf{a}\cos\phi + \mathbf{b}\sin\phi) = 0$$

is equivalent to the three relations

$$\mathbf{a}\cdot\nabla\mathbf{t}\cdot\mathbf{a} = 0, \quad \mathbf{b}\cdot\nabla\mathbf{t}\cdot\mathbf{b} = 0,$$

and
$$\mathbf{a}\cdot(\nabla\mathbf{t}\times\mathbf{t})\cdot\mathbf{a} - \mathbf{b}\cdot(\nabla\mathbf{t}\times\mathbf{t})\cdot\mathbf{b} = 0.$$

From the last of these it follows that the moments in the directions of $\mathbf{a}$ and $\mathbf{b}$ are equal. Since this must hold for all pairs of perpendicular directions in the normal plane, the conic (11) must be a circle, real or imaginary, and the moment must be the same for all directions in that plane, having the value $\frac{1}{2}\mathbf{t}\cdot\operatorname{rot}\mathbf{t}$. Thus at a point on this limit-striction surface, the moment of the family of curves on any surface of the congruence has this same value. The directions of zero moment in the normal plane are therefore minimal. Also in the particular case of a rectilinear congruence, the second

curvature K has the same value at P for all the ruled surfaces of the congruence, being equal to $-\frac{1}{4}(\mathbf{t}\cdot\operatorname{rot}\mathbf{t})^2$; and the parameter of distribution of the ray is the same for all such surfaces.

For an isotropic congruence of curves the intersection of the limit-striction surface with any surface of the congruence is the line of striction of the family of curves on that surface; for it is the locus of points at which the tendency is zero for the direction perpendicular to the curve.

Similarly we might imagine a curvilinear congruence with the property that, at all points of the surface of normality, the cone of Malus consists of a pair of planes, one of which is the normal plane to the curve. Such points must lie also on the ultimate surface; and the coalescence of these two surfaces gives what may be called the ultimate-normality surface. As before, let $\mathbf{a}$ and $\mathbf{b}$ be perpendicular unit vectors in the normal plane. Then, if this plane is part of the cone of zero moment, we must have

$$(\mathbf{a}\cos\phi+\mathbf{b}\sin\phi)\cdot(\nabla\mathbf{t}\times\mathbf{t})\cdot(\mathbf{a}\cos\phi+\mathbf{b}\sin\phi)=0$$

for all values of ϕ. This is equivalent to the three equations

$$\mathbf{a}\cdot(\nabla\mathbf{t}\times\mathbf{t})\cdot\mathbf{a}=0,\quad \mathbf{b}\cdot(\nabla\mathbf{t}\times\mathbf{t})\cdot\mathbf{b}=0,$$

and $$\mathbf{b}\cdot\nabla\mathbf{t}\cdot\mathbf{b}-\mathbf{a}\cdot\nabla\mathbf{t}\cdot\mathbf{a}=0.$$

From the last of these it follows that the tendencies in the directions of $\mathbf{a}$ and $\mathbf{b}$ are equal; and since this is true for all pairs of perpendicular vectors, the conic (4) must be a circle, real or imaginary. Thus the tendency has the same value, $\frac{1}{2}\operatorname{div}\mathbf{t}$, for all directions in the normal plane, being also equal to the divergence, at that point, of the family of curves on any surface of the congruence through the point.

For a congruence of this nature the directions of zero tendency in the normal plane are minimal at points on the ultimate-normality surface. Also the intersection of this surface with any surface of the congruence is a line of zero moment for the family of curves on the latter; for the moment of this family is zero for the direction in the surface perpendicular to the curve.

139. Three orthogonal congruences. Finally let us consider briefly some properties of three curvilinear congruences cutting one another orthogonally at all points. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be the

unit tangents to the curves at any point, so that

$$\mathbf{a} = \mathbf{b} \times \mathbf{c}, \quad \mathbf{b} = \mathbf{c} \times \mathbf{a}, \quad \mathbf{c} = \mathbf{a} \times \mathbf{b}.$$

The total moments A, B, C of the three congruences have the values

$$A = \mathbf{a} \cdot \text{rot}\,\mathbf{a}, \quad B = \mathbf{b} \cdot \text{rot}\,\mathbf{b}, \quad C = \mathbf{c} \cdot \text{rot}\,\mathbf{c}.$$

From the last equation we have

$$\begin{aligned} C &= \mathbf{c} \cdot \text{rot}\,(\mathbf{a} \times \mathbf{b}) \\ &= \mathbf{c} \cdot (\mathbf{b} \cdot \nabla\mathbf{a} - \mathbf{a} \cdot \nabla\mathbf{b} + \mathbf{a}\,\text{div}\,\mathbf{b} - \mathbf{b}\,\text{div}\,\mathbf{a}), \end{aligned}$$

or

$$C = \mathbf{b} \cdot (\nabla\mathbf{a} \times \mathbf{a}) \cdot \mathbf{b} + \mathbf{a} \cdot (\nabla\mathbf{b} \times \mathbf{b}) \cdot \mathbf{a} \quad \text{.........(18).}$$

Also, since $\mathbf{a}$ is perpendicular to $\mathbf{b}$, it follows that

$$0 = \nabla(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot \nabla\mathbf{b} + \mathbf{b} \cdot \nabla\mathbf{a} + \mathbf{a} \times \text{rot}\,\mathbf{b} + \mathbf{b} \times \text{rot}\,\mathbf{a},$$

and consequently, on forming the scalar product with $\mathbf{b} \times \mathbf{a}$, we have

$$0 = \mathbf{a} \cdot (\nabla\mathbf{b} \times \mathbf{b}) \cdot \mathbf{a} - \mathbf{b} \cdot (\nabla\mathbf{a} \times \mathbf{a}) \cdot \mathbf{b} - B + A \text{......(19).}$$

From this equation and (18) it follows that

$$\left.\begin{aligned} \mathbf{a} \cdot (\nabla\mathbf{b} \times \mathbf{b}) \cdot \mathbf{a} &= \tfrac{1}{2}(B + C - A) \\ \mathbf{b} \cdot (\nabla\mathbf{a} \times \mathbf{a}) \cdot \mathbf{b} &= \tfrac{1}{2}(A - B + C) \end{aligned}\right\} \text{...........(20).}$$

Similarly we may show that

$$\left.\begin{aligned} \mathbf{a} \cdot (\nabla\mathbf{c} \times \mathbf{c}) \cdot \mathbf{a} &= \tfrac{1}{2}(B + C - A) \\ \mathbf{c} \cdot (\nabla\mathbf{a} \times \mathbf{a}) \cdot \mathbf{c} &= \tfrac{1}{2}(A + B - C) \end{aligned}\right\} \text{...........(21).}$$

From (20) and (21) it is clear that the moments of the second and third congruences in the direction of the first have the same value. If we write

$$2S = A + B + C,$$

we may state the above results as follows:

The moments of the second and third congruences in the direction of the first have each the value $S - A$; those of the first and third in the direction of the second have the value $S - B$, and those of the first and second in the direction of the third the value $S - C$.

An important particular case is that in which two of the congruences are normal, say the first and second. Then A and B both vanish; and the curves of the third congruence are the lines of intersection of two orthogonal families of surfaces, whose unit normals are $\mathbf{a}$ and $\mathbf{b}$ respectively. The moment of the third congruence in either of the directions $\mathbf{a}$ or $\mathbf{b}$ is equal to $\frac{1}{2}C$. Thus:

If the curves of a congruence are the lines of intersection of two orthogonal families of surfaces, the moments of the congruence in the directions of the normals to these surfaces are equal to each other and to half the total moment of the congruence.

The theorem just enunciated has several known theorems as immediate corollaries. The moment of the congruence in the direction of $\mathbf{b}$ is the moment of the family of curves on the surface whose normal is $\mathbf{a}$; and this vanishes identically only when these curves are lines of curvature on the surface (Art. 10). Similarly the moment of the congruence in the direction of $\mathbf{a}$ is that of the family of curves on the surface whose normal is $\mathbf{b}$. Hence the theorem that if the curves of intersection are lines of curvature on one of the orthogonal families of surfaces, they are lines of curvature also on the other*. In that case C is zero, and the curves of intersection constitute a normal congruence. Thus we have Darboux's theorem that there is a third family of surfaces orthogonal to the first two†. Conversely, if C is zero, the moment of the family of curves on each of the orthogonal surfaces is zero, and the curves are therefore lines of curvature. Hence we have Dupin's theorem that the curves of intersection of the surfaces of a triply orthogonal system are lines of curvature on these surfaces‡.

140. Congruence of lines of equidistance§. To illustrate the preceding theory let us consider the congruence of lines of equidistance of a given family of surfaces. It was shown in Arts. 56, 57 that, if $\mathbf{n}$ is the unit normal for the family of surfaces and κ the curvature of their orthogonal trajectories,

$$\kappa^2 = (\operatorname{rot} \mathbf{n})^2,$$

and the unit tangent $\mathbf{t}$ to the line of equidistance through any point is given by

$$\mathbf{t} = \rho \operatorname{rot} \mathbf{n} \qquad \text{.......................(22)},$$

ρ being the radius of curvature of the orthogonal trajectory of the surfaces, and therefore the reciprocal of κ. The *surface of striction*

* Vol. I, Art. 51.

† Vol. I, Art. 112.

‡ Vol. I, Art. 109.

§ See a paper by the author, "On the Lines of Equidistance of a Family of Surfaces," *Liouville's Journal*, loc. cit.

of the congruence is given by

$$0 = \operatorname{div}(\rho \operatorname{rot} \mathbf{n}) = \nabla\rho \cdot \operatorname{rot} \mathbf{n} \quad \ldots\ldots\ldots\ldots\ldots(23).$$

Thus the surface of striction of the congruence of lines of equidistance is the locus of points at which these curves are tangent to a surface $\rho = \text{const.}$ Or (23) may be interpreted that the surface of striction is the locus of points at which the magnitude of rot $\mathbf{n}$ is stationary for displacement along the line of equidistance.

The *limit surface* of the congruence is given by (7′). Now, in virtue of (22), it is easily verified that

$$\mathbf{t} \operatorname{div} \mathbf{t} + \mathbf{t} \times \operatorname{rot} \mathbf{t} = \kappa\nabla\rho + \rho^2 (\operatorname{rot} \mathbf{n}) \times \operatorname{rot} \operatorname{rot} \mathbf{n}.$$

But $\qquad (\operatorname{rot} \mathbf{n}) \times \operatorname{rot} \operatorname{rot} \mathbf{n} = \frac{1}{2}\nabla (\operatorname{rot} \mathbf{n})^2 - (\operatorname{rot} \mathbf{n}) \cdot \nabla \operatorname{rot} \mathbf{n},$

and therefore

$$\mathbf{t} \operatorname{div} \mathbf{t} + \mathbf{t} \times \operatorname{rot} \mathbf{t} = -\rho^2 (\operatorname{rot} \mathbf{n}) \cdot \nabla \operatorname{rot} \mathbf{n}.$$

Hence the equation of the limit surface reduces to

$$\operatorname{div} [\rho^2 (\operatorname{rot} \mathbf{n}) \cdot \nabla \operatorname{rot} \mathbf{n}] = 0 \quad \ldots\ldots\ldots\ldots\ldots(24).$$

If d/ds denotes differentiation in the direction of the line of equidistance, this equation may also be expressed

$$\operatorname{div} \left(\rho \frac{d}{ds} \operatorname{rot} \mathbf{n}\right) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(24').$$

The *surface of normality* of the congruence has for its equation

$$0 = \mathbf{t} \cdot \operatorname{rot} \mathbf{t} = \rho \operatorname{rot} \mathbf{n} \cdot (\nabla\rho \times \operatorname{rot} \mathbf{n} + \rho \operatorname{rot} \operatorname{rot} \mathbf{n}),$$

that is $\qquad (\operatorname{rot} \mathbf{n}) \cdot \operatorname{rot} \operatorname{rot} \mathbf{n} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(25),$

and the congruence will be *normal* if this equation is satisfied identically. When this is the case, the *surfaces orthogonal to the lines of equidistance* have a first curvature J' given by

$$J' = -\operatorname{div} (\rho \operatorname{rot} \mathbf{n}) = -\nabla\rho \cdot \operatorname{rot} \mathbf{n} \ldots\ldots\ldots\ldots(26),$$

and a second curvature K' given by

$$\left.\begin{aligned} 2K' &= \operatorname{div} (\mathbf{t} \operatorname{div} \mathbf{t} + \mathbf{t} \times \operatorname{rot} \mathbf{t}) \\ &= -\operatorname{div} [\rho^2 (\operatorname{rot} \mathbf{n}) \cdot \nabla \operatorname{rot} \mathbf{n}] \\ &= -\operatorname{div} \left(\rho \frac{d}{ds} \operatorname{rot} \mathbf{n}\right) \end{aligned}\right\} \ldots\ldots\ldots\ldots(27).$$

The equation (14) of the *ultimate surface* of the congruence may also be expressed in terms of ρ and $\mathbf{n}$.

141. Small deformation of a congruence of curves*. As a final illustration let us suppose that a congruence of curves with unit tangent $\mathbf{t}$ suffers a small deformation, so that the point of the curve originally at $P(\mathbf{r})$ undergoes a small displacement $\mathbf{s}$, its new position being $P'(\mathbf{r}+\mathbf{s})$. On any one curve $\mathbf{s}$ is a function of a single parameter; but on the congruence it is a point-function in space. The extension modulus ϵ for the curve is given by

$$\epsilon = \mathbf{t}\cdot\mathbf{s}' = \mathbf{t}\cdot\nabla\mathbf{s}\cdot\mathbf{t} \quad\ldots\ldots\ldots\ldots(28),$$

∇ being the three-parametric operator for space. By the same argument as in Art. 102, the unit tangent $\mathbf{t}_1$ to the deformed curve at P' is

$$\mathbf{t}_1 = (1-\epsilon)\,\mathbf{t} + \mathbf{t}\cdot\nabla\mathbf{s},$$

and the unit tangent $\bar{\mathbf{t}}$ to the deformed curve through P is given by

$$\bar{\mathbf{t}} = (1-\epsilon)\,\mathbf{t} + \mathbf{t}\cdot\nabla\mathbf{s} - \mathbf{s}\cdot\nabla\mathbf{t} \quad\ldots\ldots\ldots\ldots(29),$$

which may also be expressed

$$\bar{\mathbf{t}} = (1-\epsilon)\,\mathbf{t} + \operatorname{rot}(\mathbf{s}\times\mathbf{t}) - \mathbf{s}\operatorname{div}\mathbf{t} + \mathbf{t}\operatorname{div}\mathbf{s}\ldots\ldots(29').$$

The surface of striction of the deformed congruence is given by $\operatorname{div}\bar{\mathbf{t}} = 0$, which, in virtue of (29′), may be expanded in the form

$$(1-\epsilon)\operatorname{div}\mathbf{t} - \mathbf{t}\cdot\nabla\epsilon - \mathbf{s}\cdot\nabla\operatorname{div}\mathbf{t} + \mathbf{t}\cdot\nabla\operatorname{div}\mathbf{s} = 0\ldots\ldots(30),$$

since the divergence of the rotation of a vector vanishes identically. Similarly the equation of the limit surface and the condition of normality for the deformed congruence may be expanded in terms of $\mathbf{t}$, $\mathbf{s}$ and ϵ.

* *Bull. Amer. Math. Soc.*, Jan.–Feb. 1927, p. 62.

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